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NUMERATION SYSTEMS, FRACTALS AND STOCHASTIC PROCESSES

BY

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ABSTRACT

A numeration system Ω is a compactification of the set of real numbers keeping the actions of addition and positive multiplication in a natural way. That is, Ω is a compact metrizable space with $\sharp\Omega \geq 2$ to which $\mathbb{R}$ acts additively and G acts multiplicatively satisfying the distributive law, where G is a nontrivial closed multiplicative subgroup of $\mathbb{R}_+$. Moreover, the additive action is minimal and uniquely ergodic with 0-topological entropy, while the multiplication by λ has $|\log\lambda|$-topological entropy attained uniquely by the unique invariant probability measure under the additive action.

We construct Ω as above as a colored tiling space corresponding to a weighted substitution. This framework contains especially the substitution dynamical systems and β-transformation systems with periodic expansion of 1, both of which have discrete G. It also contains systems with $G = \mathbb{R}_+$. We study α-homogeneous cocycles on it with respect to the addition. They are interesting from the point of view of fractal functions or sets as well as self-similar processes. We obtain the zeta-functions of Ω with respect to the multiplication.

1. Numeration systems

By a **numeration system**, we mean a compact metrizable space Ω with at least 2 elements as follows:

1. There exists a nontrivial closed multiplicative subgroup G of $\mathbb{R}_+$ such that $(\mathbb{R}, G)$ **acts numerically** to Ω in the sense that there exist continuous mappings

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$\chi_1: \Omega \times \mathbb{R} \to \Omega$ and $\chi_2: \Omega \times G \to \Omega$, where we denote $\omega + t := \chi_1(\omega, t), \lambda\omega := \chi_2(\omega, \lambda)$, satisfying

$$\omega + 0 = \omega, \quad (\omega + t) + s = \omega + (t + s),$$
$$1\omega = \omega, \quad \tau(\lambda\omega) = (\tau\lambda)\omega,$$
$$\lambda(\omega + t) = \lambda\omega + \lambda t,$$

for any $\omega \in \Omega, t, s \in \mathbb{R}$ and $\lambda, \tau \in G$.

2. The additive action of $\mathbb{R}$ to Ω is minimal and uniquely ergodic having 0-topological entropy.

3. The multiplicative action of $\lambda(\in G)$ to Ω has $|\log\lambda|$-topological entropy. Moreover, the unique invariant probability measure under the additive action is invariant under the G-action and is the unique invariant probability measure attaining the topological entropy of the multiplication by $\lambda \neq 1$.

For general notions of dynamical systems like ergodicity, weakly mixing, topological or measure theoretical entropy etc., refer to K. Petersen [9] and P. Walters [12].

Note that if Ω is a numeration system, then Ω is a connected space with the continuum cardinality. Also, note that the multiplicative group G as above is either $\mathbb{R}_+$ or $\{\lambda^n; n \in \mathbb{Z}\}$ for some $\lambda > 1$. Moreover, the additive action is faithful, that is $\omega + t = \omega$ implies $t = 0$ for any $\omega \in \Omega$ and $t \in \mathbb{R}$.

This is because if there exist $\omega_1 \in \Omega$ and $t_1 \neq 0$ such that $\omega_1 + t_1 = \omega_1$, then take a sequence λ_n in G such that $\lambda_n \to 0$ and $\lambda_n\omega_1$ converges as $n \to \infty$. Let $\omega_\infty := \lim_{n\to\infty} \lambda_n\omega_1$. For any $t \in \mathbb{R}$, let a_n be a sequence of integers such that $a_n\lambda_n t_1 \to t$ as $a \to \infty$. By the distributive law and the continuity of the additive action, we have

$$\begin{aligned}\omega_\infty + t &= \lim_{n\to\infty} (\lambda_n\omega_1 + \lambda_n a_n t_1) \\ &= \lim_{n\to\infty} \lambda_n(\omega_1 + a_n t_1) = \lim_{n\to\infty} \lambda_n\omega_1 = \omega_\infty.\end{aligned}$$

Thus, ω_∞ becomes a fixed point with respect to the additive action which contradicts with the minimality of the additive action together with $\sharp\Omega \geq 2$.

An example of a numeration system is the set $\{0,1\}^{\mathbb{Z}}$ with the product topology divided by the closed equivalence relation $\sim$ such that

$$(\cdots\alpha_{-1}; \alpha_0, \alpha_1, \alpha_2\cdots) \sim (\cdots\beta_{-1}; \beta_0, \beta_1, \beta_2\cdots)$$

if and only if there exists $N \in \mathbb{Z} \cup \{\infty\}$ satisfying $\alpha_n = \beta_n (\forall n > N)$, $\alpha_N = \beta_N + 1$ and $\alpha_n = 0, \beta_n = 1$ $(\forall n < N)$ or the same statement with α and

β exchanged. Let $\Omega(2) := \{0,1\}^{\mathbb{Z}}/\sim$ and the equivalence class containing $(\cdots\alpha_{-1};\alpha_0,\alpha_1,\alpha_2\cdots) \in \{0,1\}^{\mathbb{Z}}$ is denoted by $\sum_{n=-\infty}^{\infty}\alpha_n 2^n \in \Omega(2)$. Then, $\Omega(2)$ is an additive topological group with the addition as follows:

$$\sum_{n=-\infty}^{\infty}\alpha_n 2^n + \sum_{n=-\infty}^{\infty}\beta_n 2^n = \sum_{n=-\infty}^{\infty}\gamma_n 2^n$$

if and only if there exists $(\cdots\eta_{-1};\eta_0,\eta_1,\eta_2\cdots) \in \{0,1\}^{\mathbb{Z}}$ satisfying

$$2\eta_{n+1} + \gamma_n = \alpha_n + \beta_n + \eta_n \quad (\forall n \in \mathbb{Z}).$$

This is isomorphic to the 2-adic **solenoidal group**, which is by definition the projective limit of the projective system θ: $\mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ with $\theta(\alpha) = 2\alpha(\alpha \in \mathbb{R}/\mathbb{Z})$.

Moreover, $\mathbb{R}$ is imbedded in $\Omega(2)$ continuously as a dense additive subgroup in the way that a nonnegative real number α is identified with $\sum_{n=-\infty}^{\infty}\alpha_n 2^n$ such that $\alpha = \sum_{n=-\infty}^{N}\alpha_n 2^n$ and $\alpha_n = 0$ $(\forall n > N)$ for some $N \in \mathbb{Z}$, while a negative real number $-\alpha$ with α as above is identified with $\sum_{n=-\infty}^{\infty}(1-\alpha_n)2^n$. Then, $\mathbb{R}$ acts additively to $\Omega(2)$ by this addition. Furthermore, $G := \{2^k; k \in \mathbb{Z}\}$ acts multiplicatively to $\Omega(2)$ by

$$2^k \sum_{n=-\infty}^{\infty}\alpha_n 2^n = \sum_{n=-\infty}^{\infty}\alpha_{n-k}2^n.$$

Thus, we have

THEOREM 1: *$\Omega(2)$ is a numeration system with $G = \{2^n; n \in \mathbb{Z}\}$.*

We can express $\Omega(2)$ in the following different way. Let us consider the set of partitions ω of the plane $\mathbb{R}^2$ with the x-y-coordinate system by rectangles of the form $[x_1,x_2) \times [y_1,y_2)$ with $x_2 - x_1 = e^{y_1}$, $y_1 \in \mathbb{Z}\log 2$ and $y_2 - y_1 = \log 2$ such that $[x_1,x_2)\times[y_1,y_2) \in \omega$ implies

$$(1) \qquad \begin{aligned} &[x_1,(x_1+x_2)/2) \times [y_1 - \log 2, y_1) \in \omega \text{ (type 0)} \\ &\text{and} \\ &[(x_1+x_2)/2, x_2) \times [y_1 - \log 2, y_1) \in \omega \text{ (type 1)}. \end{aligned}$$

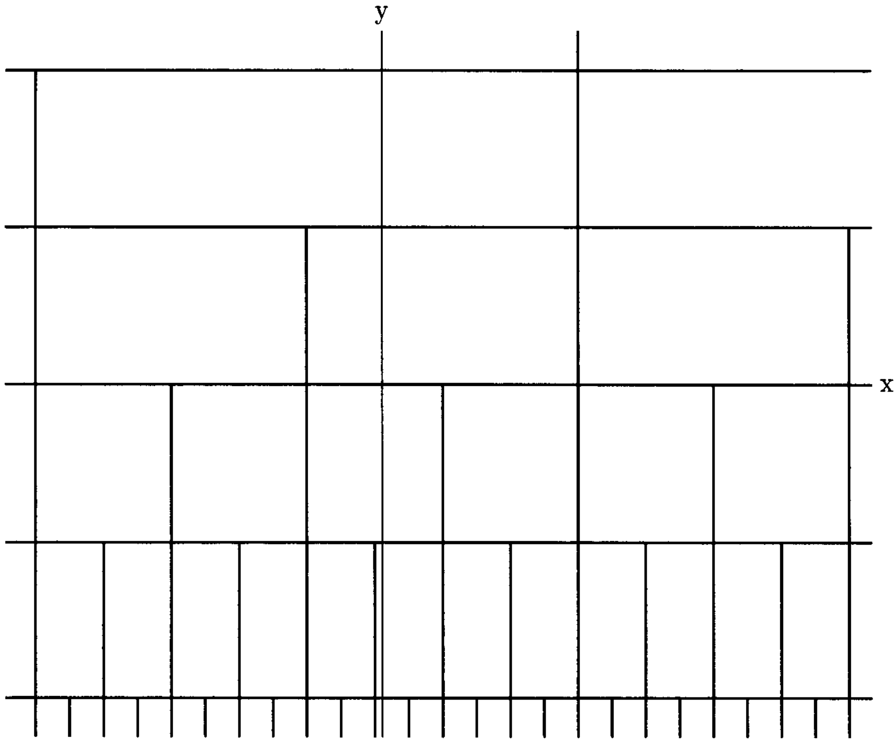

Figure 1. The tiling corresponding to $\cdots 11.010 \cdots$

We consider $\Omega(2)'$ as the set of partitions (called tilings as well) ω of $\mathbb{R}^2$ as above (see Figure 1). For $\omega \in \Omega(2)'$, let $(\ldots, \alpha_1, \alpha_0)$ be the sequence of the types defined in (1) of the sets in ω intersecting with the half line $\{0\} \times [0, \infty)$ from above and let $(\alpha_{-1}, \alpha_{-2}, \ldots)$ be the sequence of the types of the sets in ω intersecting with the half line $\{0\} \times (-\infty, 0)$. Then, ω is identified with $\sum_{n=-\infty}^{\infty} \alpha_n 2^n$.

The topology on $\Omega(2)'$ is defined so that $\omega_n \in \Omega(2)'$ converges to $\omega \in \Omega(2)'$ as $n \to \infty$ if for every $R \in \omega$, there exist $R_n \in \omega_n$ such that $\lim_{n\to\infty} \rho(R, R_n) = 0$, where ρ is the "Hausdorff metric", that is, for any subsets $R, R' \subset \mathbb{R}^2$

$$(2) \qquad \rho(R, R') := \max\{\sup_{z\in R} \inf_{z'\in R'} \| z - z' \|, \sup_{z'\in R'} \inf_{z\in R} \| z - z' \|\}$$
$$(\| z \| = \sqrt{x^2 + y^2} \text{ for } z = (x, y) \in \mathbb{R}^2).$$

Note that ρ is not a metric, but it is a pseudo-metric taking values in $[0, \infty]$ in general. It is usually considered on the family of compact sets and becomes a metric. Here, we consider it on the family of sets of type $[x_1, x_2) \times [y_1, y_2)$ and it becomes a metric. Later, we consider it on a little more general family of sets such that it is still a metric.

For $\omega \in \Omega(2)'$, $t \in \mathbb{R}$ and $\lambda \in \{2^n; n \in \mathbb{R}\}$, $\omega + t \in \Omega(2)'$ and $\lambda\omega \in \Omega(2)'$ are defined respectively as the partitions

$$\omega + t := \{[x_1 - t, x_2 - t) \times [y_1, y_2); [x_1, x_2) \times [y_1, y_2) \in \omega\}$$

and

$$\lambda\omega := \{[\lambda x_1, \lambda x_2) \times [y_1 + \log \lambda, y_2 + \log \lambda]; [x_1, x_2) \times [y_1, y_2) \in \omega\}$$

of $\mathbb{R}^2$.

Let κ: $\Omega(2)' \to \Omega(2)$ be the identification mapping defined above. Then, κ is a homeomorphism between $\Omega(2)'$ and $\Omega(2)$ such that $\kappa(\omega + t) = \kappa(\omega) + t$ and $\kappa(\lambda\omega) = \lambda\kappa(\omega)$ for any $\omega \in \Omega(2)', t \in \mathbb{R}$ and $\lambda \in \{2^n; n \in \mathbb{Z}\}$. Thus, $\Omega(2)'$ is isomorphic to $\Omega(2)$ as a numeration system and will be identified with $\Omega(2)$.

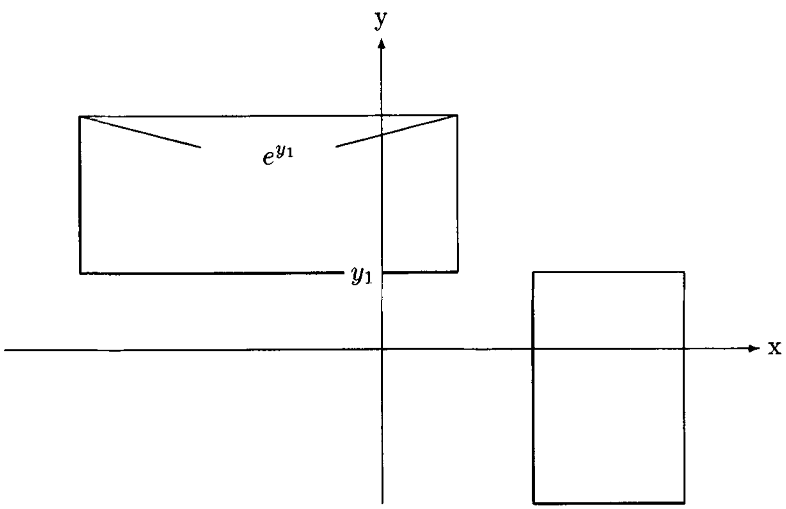

Figure 2. Admissible tiles

We generalize this construction. Let $\mathbb{A}$ be a nonempty finite set. An element in $\mathbb{A}$ is called a **color**. A rectangle $[x_1, x_2) \times [y_1, y_2)$ in $\mathbb{R}^2$ is called an **admissible tile** if $x_2 - x_1 = e^{y_1}$ is satisfied (see Figure 2). A **colored tiling** ω with colors in $\mathbb{A}$ is a mapping from $\mathrm{dom}(\omega)$ to $\mathbb{A}$, where $\mathrm{dom}(\omega)$ consists of admissible tiles which are disjoint each other and the union of which is $\mathbb{R}^2$. For $R \in \mathrm{dom}(\omega)$,

$\omega(R)$ is considered as the color painted on the admissible tile R. In other words, a colored tiling is a partition of $\mathbb{R}^2$ by admissible tiles with colors in $\mathbb{A}$. Let $\Omega(\mathbb{A})$ be the set of colored tilings with colors in $\mathbb{A}$

A topology is introduced on $\Omega(\mathbb{A})$ so that a net $\{\omega_n\}_{n\in I} \subset \Omega(\mathbb{A})$ converges to $\omega \in \Omega(\mathbb{A})$ if for every $R \in \mathrm{dom}(\omega)$, there exist $R_n \in \mathrm{dom}(\omega_n)$ $(n \in I)$ such that

$$\omega(R) = \omega_n(R_n) \text{ for any sufficiently large } n \in I \text{ and } \lim_{n\to\infty} \rho(R, R_n) = 0,$$

where ρ is the Hausdorff metric defined in (2).

For an admissible tile $R := [x_1, x_2) \times [y_1, y_2)$, $t \in \mathbb{R}$ and $\lambda \in \mathbb{R}_+$, we denote

$$\begin{aligned} R + t &:= [x_1 - t, x_2 - t) \times [y_1, y_2), \\ \lambda R &:= [\lambda x_1, \lambda x_2) \times [y_1 + \log \lambda, y_2 + \log \lambda). \end{aligned}$$

Note that they are also admissible tiles.

For $\omega \in \Omega(\mathbb{A})$, $t \in \mathbb{R}$ and $\lambda \in \mathbb{R}_+$, we define $\omega + t \in \Omega(\mathbb{A})$ and $\lambda\omega \in \Omega(\mathbb{A})$ as follows:

$$\begin{aligned} \mathrm{dom}(\omega + t) &:= \{R + t; R \in \mathrm{dom}(\omega)\}, \\ (\omega + t)(R + t) &:= \omega(R) \text{ for any} R \in \mathrm{dom}(\omega), \\ \mathrm{dom}(\lambda\omega) &:= \{\lambda R; R \in \mathrm{dom}(\omega)\}, \\ (\lambda\omega)(\lambda R) &:= \omega(R) \text{ for any} R \in \mathrm{dom}(\omega). \end{aligned} \tag{3}$$

Thus, $(\mathbb{R}, \mathbb{R}_+)$ acts numerically to $\Omega(\mathbb{A})$. We construct compact metrizable subspaces of $\Omega(\mathbb{A})$ corresponding to weighted substitutions which are numeration systems. Though $\sharp\mathbb{A} \geq 2$ is assumed in [7], we consider the case $\sharp\mathbb{A} = 1$ as well.

2. Remarks on the notations

In this paper, the notations are changed on a large scale from the previous papers [4], [7] and [8] of the author. The main changes are as follows:

(1) The roles of x-axis and y-axis for colored tilings are exchanged. The additive action corresponds to the translation along the x-axis while the multiplicative action corresponds to the translation along the y-axis together with horizontal homothety. Tiles are denoted by $[x_1, x_2) \times [y_1, y_2)$ instead of $(a, b] \times [c, d)$. The admissibility of tiles is expressed as $x_2 - x_1 = e^{y_1}$ instead of $d - c = e^{-b}$.

(2) The set of colors is denoted by $\mathbb{A}$ instead of Σ. Colors are denoted by a, a', a_i (etc.) instead of $\sigma, \sigma', \sigma_i$ (etc.).

(3) The weighted substitution is denoted by (σ, τ) instead of (φ, η).

(4) Admissible tiles are denoted by R, R', R_i, R^i (etc.) instead of S, S', S_i, S^i (etc.).

(5) The terminology "primitive" for substitutions is used instead of "mixing" in the previous papers.

3. Weighted substitutions

A **substitution** σ on a set $\mathbb{A}$ is a mapping $\mathbb{A} \to \mathbb{A}^+$, where $\mathbb{A}^+ = \bigcup_{\ell=1}^{\infty} \mathbb{A}^\ell$. For $\xi \in \mathbb{A}^+$, we denote $|\xi| := \ell$ if $\xi \in \mathbb{A}^\ell$, and ξ with $|\xi| = \ell$ is usually denoted by $\xi_0\xi_1 \cdots \xi_{\ell-1}$ with $\xi_i \in \mathbb{A}$. We can extend σ to be a homomorphism $\mathbb{A}^+ \to \mathbb{A}^+$ as follows:

$$\sigma(\xi) := \sigma(\xi_0)\sigma(\xi_1) \cdots \sigma(\xi_{\ell-1}),$$

where $\xi \in \mathbb{A}^\ell$ and the right-hand side is the concatenations of $\sigma(\xi_i)$'s. We can define $\sigma^2, \sigma^3, \ldots$ as the compositions of σ: $\mathbb{A}^+ \to \mathbb{A}^+$.

A **weighted substitution** (σ, τ) on $\mathbb{A}$ is a mapping $\mathbb{A} \to \mathbb{A}^+ \times (0,1)^+$ such that $|\sigma(a)| = |\tau(a)|$ and $\sum_{i<|\tau(a)|} \tau(a)_i = 1$ for any $a \in \mathbb{A}$. Note that σ is a substitution on $\mathbb{A}$. We define τ^n: $\mathbb{A} \to (0,1)^+ (n = 2, 3, \ldots)$ inductively by

$$\tau^n(a)_k = \tau(a)_i \tau^{n-1}(\sigma(a)_i)_j$$

for any $a \in \mathbb{A}$ and i, j, k with

$$0 \le i < |\sigma(a)|, \quad 0 \le j < |\sigma^{n-1}(\sigma(a)_i)|, \quad k = \sum_{h<i} |\sigma^{n-1}(\sigma(a)_h)| + j.$$

Then, (σ^n, τ^n) is also a weighted substitution for $n = 2, 3, \ldots$.

A substitution σ on $\mathbb{A}$ is called **primitive** if there exists a positive integer n such that for any $a, a' \in \mathbb{A}$, $\sigma^n(a)_i = a'$ holds for some i with $0 \le i < |\sigma^n(a)|$.

For a weighted substitution (σ, τ) on $\mathbb{A}$, we always assume that

$$\text{the substitution } \sigma \text{ is primitive.} \tag{4}$$

We define the **base set** $B(\sigma, \tau)$ as the closed, multiplicative subgroup of $\mathbb{R}_+$ generated by the set

$$\left\{ \begin{matrix} \tau^n(a)_i; & a \in \mathbb{A}, & n = 0, 1, \ldots \text{ and } 0 \le i < |\sigma^n(a)| \\ & & \text{such that } \sigma^n(a)_i = a \end{matrix} \right\}.$$

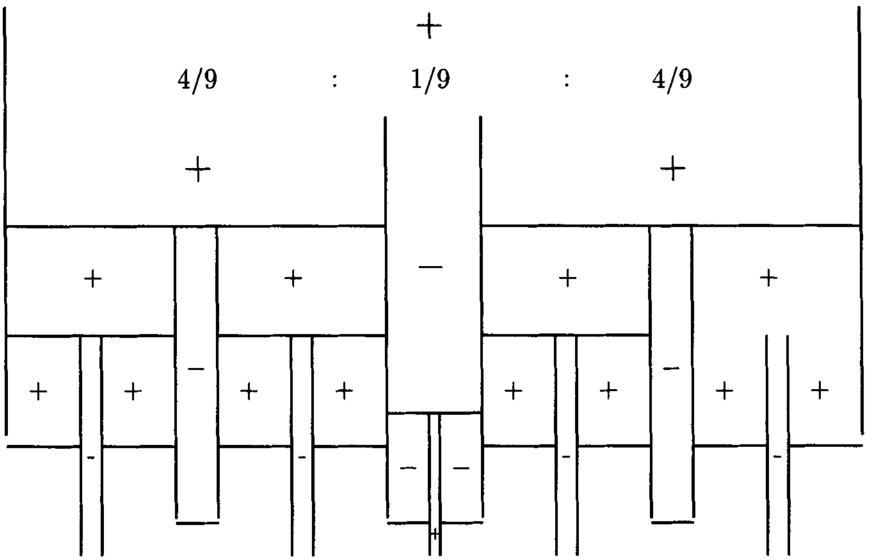

Figure 3. The weighted substitution in Example 1

Example 1: Let $\mathbb{A} = \{+,-\}$ and (σ,τ) be a weighted substitution such that

$$+ \to (+,4/9)(-,1/9)(+,4/9),$$
$$- \to (-,4/9)(+,1/9)(-,4/9),$$

where we express a weighted substitution (σ,τ) by

$$a \to (\sigma(a)_0, \tau(a)_0)(\sigma(a)_1, \tau(a)_1)\cdots \quad (a \in \mathbb{A}).$$

Then, $4/9 \in B(\sigma,\tau)$ since $\sigma(+)_0 = +$ and $\tau(+)_0 = 4/9$. Moreover, $1/81 \in B(\sigma,\tau)$ since $\sigma^2(+)_4 = +$ and $\tau^2(+)_4 = 1/81$. Since $4/9$ and $1/81$ do not have a common multiplicative base, we have $B(\sigma,\tau) = \mathbb{R}_+$. This weighted substitution is discussed in the following sections. The repetition of this weighted substitution starting at $+$ is shown in Figure 3, where $+$, $-$ are represented as the color of tiles. The substituted word of a color is represented as the sequence of colors of the connected tiles below in order from the left. The horizontal sizes of tiles are proportional to the weights and the vertical sizes are the minus of the logarithm of the weights.

Let $G := B(\sigma,\tau)$. Then, there exists a function $g\colon \mathbb{A} \to \mathbb{R}_+$ such that

$$g(\sigma(a)_i)G = g(a)\tau(a)_i G \tag{5}$$

for any $a \in \mathbb{A}$ and $0 \le i < |\sigma(a)|$. Note that if $G = \mathbb{R}_+$, then we can take $g \equiv 1$. In the other case, we can define g by $g(a_0) = 1$ and $g(a) := \tau^n(a_0)_i$ for some n and i such that $\sigma^n(a_0)_i = a$, where a_0 is any fixed element in $\mathbb{A}$.

Let (σ, τ) be a weighted substitution satisfying (4). Let $G = B(\sigma, \tau)$. Let g satisfy (5). Let $\Omega(\sigma, \tau, g)'$ be the set of all elements ω in $\Omega(\mathbb{A})$ such that for any $[x_1, x_2) \times [y_1, y_2) \in \mathrm{dom}(\omega)$ with $\omega([x_1, x_2) \times [y_1, y_2)) = a$, we have

(I) $e^{y_1} \in g(a)G$, and

(II) $R^i \in \mathrm{dom}(\omega)$ and $\omega(R^i) = \sigma(a)_i$ hold for $i = 0, 1, \ldots, |\sigma(a)| - 1$, where

$$R^i := [x_1 + (x_2 - x_1)\sum_{j=0}^{i-1} \tau(a)_j, x_1 + (x_2 - x_1)\sum_{j=0}^{i} \tau(a)_j) \times [y_1 + \log \tau(a)_i, y_1).$$

A vertical line $\gamma := \{x\} \times (-\infty, \infty)$ is called a **separating line** of $\omega \in \Omega(\sigma, \tau, g)'$ if for any $R \in \mathrm{dom}(\omega)$, $R^\circ \cap \gamma = \emptyset$, where R° denotes the set of inner points of R. Let $\Omega(\sigma, \tau, g)''$ be the set of all $\omega \in \Omega(\sigma, \tau, g)'$ which do not have a separating line and $\Omega(\sigma, \tau, g)$ be the closure of $\Omega(\sigma, \tau, g)''$. Then, $(\mathbb{R}, G)$ acts to $\Omega(\sigma, \tau, g)$ numerically. We usually denote $\Omega(\sigma, \tau, 1)$ simply by $\Omega(\sigma, \tau)$.

Remark 1 [7]*:* A nontrivial primitive substitution $\sigma : \mathbb{A} \to \mathbb{A}^+$, where "nontrivial" means $\sum_{a \in \mathbb{A}} |\sigma(a)| \ge 2$, is considered as a weighted substitution in a canonical way. Let

$$M := (\sharp\{0 \le i < |\sigma(a)|; \sigma(a)_i = a'\})_{a, a' \in \mathbb{A}}$$

be the associate matrix. Let λ be the maximum eigen-value of M and $\xi := (\xi_a)_{a \in \mathbb{A}}$ be a positive column vector such that $M\xi = \lambda\xi$. Define weight τ by

$$\tau(a)_i = \frac{\xi_{\sigma(a)_i}}{\lambda \xi_a},$$

which is called the **natural weight** of σ. Thus, we get a weighted substitution (σ, τ) which admits weight 1. We modify (σ, τ) if necessary in the following way. If there exists $a \in \mathbb{A}$ with $|\sigma(a)| = 1$, so that $a \to (a', 1)$ is a part of (σ, τ), then we replace all the occurrences of a in the right hand side of "$\to$" by a' and remove a from $\mathbb{A}$ together with the rule $a \to (a', 1)$ from (σ, τ). We continue this process until no $a \in \mathbb{A}$ satisfies $|\sigma(a)| = 1$. After that, if there exist $a, a' \in \mathbb{A}$ such that $(\sigma(a), \tau(a)) = (\sigma(a'), \tau(a'))$, then we identify them.

For example, the **2-adic expansion substitution** $1 \to 12$, $2 \to 12$ corresponds to the weighted substitution $1 \to (1, 1/2)(1, 1/2)$. The **Thue–Morse substitution** $1 \to 12$, $2 \to 21$ corresponds to the weighted substitution $1 \to$

$(1,1/2)(2,1/2)$, $2 \to (2,1/2)(1,1/2)$. The **Fibonacci substitution** $1 \to 12$, $2 \to 1$ corresponds to the weighted substitution $1 \to (1,\lambda^{-1})(1,\lambda^{-2})$, where $\lambda = (1+\sqrt{5})/2$.

The weighted substitution (σ,τ) obtained in this way satisfies $B(\sigma,\tau) = \{\lambda^n; n \in \mathbb{Z}\}$ and that g in (5) can be defined by $g(a) = \xi_a (a \in \mathbb{A})$. Dynamical systems coming from substitutions are discussed by many authors (see [2], for example). Our weighted substitutions are a generalization of them.

Let (σ,τ) be a weighted substitution on $\mathbb{A}$ satisfying (4). Let g satisfy (5). Consider $\Omega(\sigma,\tau,g)$. We call the tile R^i in (II) the i-th **child** of the tile $[x_1,x_2) \times [y_1,y_2)$ (in ω), and the tile $[x_1,x_2) \times [y_1,y_2)$ the **mother** of R^i. If R_{j+1} is a child of R_j for $j = 0,1,\ldots,k-1$, then the tile R_k is called a k-th **descendant** of the tile R_0. If R_k is the i-th tile among the set of the k-th descendants of R_0 counting as $0,1,2,\ldots$ from the left, we call R_k the (k,i)-**descendant** of the tile R_0. In this case, we also say that R_0 is the k-th **ancestor** of R_k.

THEOREM 2: *The space $\Omega(\sigma,\tau,g)$ is a numeration system with $G = B(\sigma,\tau)$.*

Proof: The properties 1 and 2 in the definition of numeration system are already proved in Theorem 3 in [7]. We prove the property 3.

Let $\Omega := \Omega(\sigma,\tau,g)$. The family of sets

$$\begin{gathered} U_\epsilon(R_1,\ldots,R_K;a_1,\ldots,a_K) := \\ \{\omega \in \Omega;\ \text{there exists } R'_k \in \mathrm{dom}(\omega) \text{ such that} \\ \omega(R'_k) = a_k \text{ and } \rho(R_k,R'_k) < \epsilon \text{ for } k = 1,\ldots,K\}, \end{gathered}$$

where $\epsilon > 0$, $K = 1,2,\ldots$ and for $k = 1,\ldots,K$, $a_k \in \mathbb{A}$ and R_k is an admissible tile such that the above set is nonempty, is an open base of the space Ω ([7]).

For any $N \in \mathbb{R}$, define $d_N\colon \Omega \times \Omega \to [0,\infty)$ as follows:

$$\begin{aligned} &d_N(\omega,\omega') := \\ &\max\left\{ \sup_{R\in\mathrm{dom}(\omega)} \inf_{\substack{R'\in\mathrm{dom}(\omega') \\ \omega(R)=\omega'(R')}} \rho_N(R,R'),\ \sup_{R'\in\mathrm{dom}(\omega')} \inf_{\substack{R\in\mathrm{dom}(\omega) \\ \omega(R)=\omega'(R')}} \rho_N(R,R') \right\}, \end{aligned}$$

where for any sets R, R' of type $[x_1,x_2) \times [y_1,y_2)$,

$$\begin{aligned} &\rho_N(R,R') := \rho((R\cap D_N)\cup\partial D_N, (R'\cap D_N)\cup\partial D_N), \\ (7)\qquad &D_N := \{(x,y)\in\mathbb{R}^2; -e^N \le x \le e^N \text{ and } y \le N\}, \\ &\partial D_N \text{ is the boundary of } D_N. \end{aligned}$$

LEMMA 1: *For any $\omega, \omega' \in \Omega$, let $R \in \mathrm{dom}(\omega)$ and $R' \in \mathrm{dom}(\omega')$ satisfy $\omega(R) = \omega'(R') := a$. For any $k = 1, 2, \dots$ and i with $0 \le i < |\sigma^k(a)|$, let S be the (k,i)-descendant of R in ω and S' be the (k,i)-descendant of R' in ω'. Then we have $\omega(S) = \omega'(S')$ and $\rho(S, S') \le \rho(R, R')$.*

Proof: Clear from the definition of $\Omega(\sigma, \tau, g)$.

LEMMA 2: *For any N, $d_N\colon \Omega \times \Omega \to [0, \infty)$ is continuous and a pseudo-metric on Ω such that $d_N(\omega, \omega') \le d_{N'}(\omega, \omega')$ for any $N < N'$ and $\omega, \omega' \in \Omega$. Moreover, $d_N(\omega, \omega') = 0$ for any N implies that $\omega = \omega'$. Hence, ω_n converges to ω (as $n \to \infty$) if and only if $\lim_{n\to\infty} d_N(\omega_n, \omega) = 0$ for any N.*

Proof: The fact that d_N is a pseudo-metric follows from the fact that ρ_N is a pseudo-metric. Therefore, to prove the continuity of d_N, it is sufficient to prove that $\lim_{n\to\infty} d_N(\omega_n, \omega) = 0$ if $\omega_n \to \omega$ (as $n \to \infty$). Assume that $\omega_n \to \omega$. Take any sufficiently small $\epsilon > 0$. Let $\{R_1, \dots, R_K\}$ be the set of tiles in $\mathrm{dom}(\omega)$ which intersects with $[-e^N, e^N] \times \{N\}$. Let

$$U := U_\epsilon(R_1, \dots, R_K; \omega(R_1), \dots, \omega(R_K)).$$

Since U is a neighborhood of ω, there exists n_0 such that $\omega_n \in U$ for any $n \ge n_0$. Then by Lemma 1, we have

$$\sup_{R \in \mathrm{dom}(\omega)} \inf_{\substack{R' \in \mathrm{dom}(\omega_n) \\ \omega(R) = \omega_n(R')}} \rho_N(R, R') < \epsilon. \tag{8}$$

Take any $R' \in \mathrm{dom}(\omega_n)$ which intersects with

$$[-e^N + 4\epsilon, e^N - 4\epsilon] \times \{N - 4\epsilon\}.$$

Then, there exists $z' \in R' \cap D_N$ such that

$$\| z' - z'' \| \ge 2\epsilon \tag{9}$$

for any $z'' \in (D_N \setminus R') \cup \partial D_N$. Suppose that

$$\inf_{\substack{R \in \mathrm{dom}(\omega) \\ \omega(R) = \omega'(R')}} \rho_N(R, R') \ge 2\epsilon. \tag{10}$$

Let $R \in \mathrm{dom}(\omega)$ satisfy $z' \in R$. Let $R'' \in \mathrm{dom}(\omega_n)$ attains the "inf" in (8) for this R. Then by (8) and (10), R'' is different from R'. It follows from (9) that $\rho_N(R, R'') \ge 2\epsilon$, which is a contradiction. Thus,

$$\inf_{\substack{R \in \mathrm{dom}(\omega) \\ \omega(R) = \omega_n(R')}} \rho_N(R, R') < 2\epsilon$$

for any $R' \in \mathrm{dom}(\omega_n)$ which intersects with

$$[-e^N + 4\epsilon, e^N - 4\epsilon] \times \{N - 4\epsilon\}.$$

It follows from Lemma 1 that

$$\sup_{R' \in \mathrm{dom}(\omega')} \inf_{\substack{R \in \mathrm{dom}(\omega) \\ \omega(R) = \omega'(R')}} \rho_N(R, R') < 4\epsilon.$$

Thus, $d_N(\omega_n, \omega) < 4\epsilon$, which implies the continuity of d_N.

That $d_N(\omega, \omega') \leq d_{N'}(\omega, \omega')$ for any $N < N'$ and $\omega, \omega' \in \Omega$ is clear from the definition of d_N. If $\omega \neq \omega'$, then there exists a tile $R \in \mathrm{dom}(\omega)$ such that there does not exist a tile $R' \in \mathrm{dom}(\omega')$ with $R = R'$ and $\omega(R) = \omega'(R')$. Let N satisfy that $D_{N-1} \supset R$. Suppose that $d_N(\omega, \omega') = 0$. Then, there exists a tile $R' \in \mathrm{dom}(\omega')$ with $(R' \cap D_N) \cup \partial D_N = (R \cap D_N) \cup \partial D_N$ and $\omega(R) = \omega'(R')$. Since $D_{N-1} \supset R$ and R' is connected, this implies that $R' = R$, which contradicts the assumption on R. Hence, $d_N(\omega, \omega') > 0$ follows. Thus, $\omega = \omega'$ if and only if $d_N(\omega, \omega') = 0$ for any N. ∎

For $\omega, \omega' \in \Omega$, let

$$d(\omega, \omega') := \inf\{\eta > 0; d_{1/\eta}(\omega, \omega') \leq \eta\}. \tag{11}$$

LEMMA 3: *The above d is a metric on Ω which is consistent with the topology.*

Proof: Take arbitrary $\omega, \omega', \omega'' \in \Omega$. Since $d_{e^{-1}}(\omega, \omega') \leq d_1(\omega, \omega') \leq e$ by (6), (7), we have $d(\omega, \omega') \leq e$.

Assume that $d(\omega, \omega') < \eta$ and $d(\omega', \omega'') < \zeta$. Since by Lemma 2 and (11), $d_{1/\eta}(\omega, \omega') < \eta$ and $d_{1/\zeta}(\omega, \omega') < \zeta$, we have

$$\begin{aligned} d_{1/(\eta+\zeta)}(\omega, \omega'') &\leq d_{1/(\eta+\zeta)}(\omega, \omega') + d_{1/(\eta+\zeta)}(\omega', \omega'') \\ &\leq d_{1/\eta}(\omega, \omega') + d_{1/\zeta}(\omega', \omega'') < \eta + \zeta. \end{aligned}$$

Hence, $d(\omega, \omega'') \leq \eta + \zeta$, which implies $d(\omega, \omega'') \leq d(\omega, \omega') + d(\omega', \omega'')$.

Assume that $\omega_n \to \omega$ in Ω as $n \to \infty$. Then, by Lemma 2, $d_N(\omega_n, \omega) \to 0$ for any N. Hence, $\limsup_{n\to\infty} d(\omega_n, \omega) \leq 1/N$ for any $N > 0$. Thus, $\lim_{n\to\infty} d(\omega_n, \omega) = 0$.

Conversely, assume that $\lim_{n\to\infty} d(\omega_n, \omega) = 0$. Then, for any $M > 0$, $d_M(\omega_n, \omega) \leq 1/M$ holds for any sufficiently large n. Hence for any $N > 0$, we have

$$\limsup_{n\to\infty} d_N(\omega_n, \omega) \leq \lim_{M\to\infty} \limsup_{n\to\infty} d_M(\omega_n, \omega) \leq \lim_{M\to\infty} 1/M = 0.$$

Thus, $\omega_n \to \omega$ by Lemma 2. ∎

LEMMA 4: *For any open cover $\mathcal{U}$ of Ω, there exist N and $\epsilon > 0$ such that $\mathcal{V}(N,\epsilon)$ is a refinement of $\mathcal{U}$, where we denote*

$$\mathcal{V}(N,\epsilon) := \{\{\omega' \in \Omega; d_N(\omega,\omega') < \epsilon\}; \omega \in \Omega\}.$$

Proof: By the Lebesgue Covering Lemma, there exists $\epsilon > 0$ such that $\{\{\omega' \in \Omega; d(\omega,\omega') < 2\epsilon\}; \omega \in \Omega\}$ is a refinement of $\mathcal{U}$. Take N with $N > 1/\epsilon$. Take any $V := \{\omega' \in \Omega; d_N(\omega,\omega') < \epsilon\} \in \mathcal{V}(N,\epsilon)$ and $\omega' \in V$. Since $d_{1/\epsilon}(\omega,\omega') \le d_N(\omega,\omega') < \epsilon$, $d(\omega,\omega') \le \epsilon < 2\epsilon$ holds. This implies that $V \subset \{\omega' \in \Omega; d(\omega,\omega') < 2\epsilon\}$ and that $\mathcal{V}(N,\epsilon)$ is a refinement of $\{\{\omega' \in \Omega; d(\omega,\omega') < 2\epsilon\}; \omega \in \Omega\}$. Thus, $\mathcal{V}(N,\epsilon)$ is a refinement of $\mathcal{U}$. ∎

LEMMA 5: *$d_N(\omega,\omega+t) \le |t|$ holds for any $\omega \in \Omega$, $t \in \mathbb{R}$ and N. Moreover, $d_N(\omega,\omega+t) = |t|$ holds for any $\omega \in \Omega$, $t \in \mathbb{R}$ and $N > 1$ such that $|t| \le C_0/2$, where*

$$\begin{aligned} C_1 &:= \min\{\tau(a)_i; a \in \mathbb{A}, 0 \le i < |\tau(a)|\}, \\ C_2 &:= \max\{\tau(a)_i; a \in \mathbb{A}, 0 \le i < |\tau(a)|\}, \\ C_0 &:= C_1 \wedge (-\log C_2). \end{aligned} \tag{12}$$

Proof: It is clear that $d_N(\omega,\omega+t) \le |t|$ for any $\omega \in \Omega$, $t \in \mathbb{R}$ and N. For any $\omega \in \Omega$, take the tile $R \in \mathrm{dom}(\omega)$ with $(0,0) \in R$. Let $N > 1$. Then, $\rho_N(R,R') \ge C_0$ holds for any other tile $R' \in \mathrm{dom}(\omega)$. Therefore, $\rho_N(R+t,R') \ge C_0 - |t|$ holds for any $t \in \mathbb{R}$ with $|t| \le C_0/2$ while $\rho_N(R+t,R) = |t|$. Thus, we have $d_N(\omega,\omega+t) \ge |t|$ for any $\omega \in \Omega$ and $t \in \mathbb{R}$ with $|t| \le C_0/2$, which completes the proof. ∎

LEMMA 6: *The topological entropy $h(\lambda)$ of the multiplicative action $\lambda \in G$ to Ω satisfies $h(\lambda) = |\log\lambda|$.*

Proof: We may assume that $\lambda > 1$. Take any $N > 1$ and $\epsilon > 0$ such that $2\epsilon\lambda < C_0/2$. Take any $n = 1, 2, \ldots$.

For any $s, t \in \mathbb{R}$ with $2\epsilon\lambda^{-n+1} \le |s-t| < C_0/2$, there exists $i = 0, 1, \ldots, n-1$ such that $2\epsilon \le \lambda^i|s-t| < C_0/2$. For any $\omega \in \Omega$ and $s, t \in \mathbb{R}$ with $2\epsilon \le \lambda^i|s-t| < C_0/2$, $\omega + s$ and $\omega + t$ do not belong to the same open set in $\lambda^{-i}\mathcal{V}(N,\epsilon)$ since

$$d_N(\lambda^i(\omega+s), \lambda^i(\omega+t)) = \lambda^i|s-t| \ge 2\epsilon$$

by Lemma 5. This implies that any 2 elements in the set

$$\{\omega, \omega+\delta, \omega+2\delta, \ldots, \omega+(L-1)\delta\}$$

do not belong to the same open set in $\vee_{i=0}^{n-1}\lambda^{-i}\mathcal{V}(N,\epsilon)$, where $\delta := 2\epsilon\lambda^{-n+1}$, $L := \lfloor C_0/2\delta \rfloor$ and $\vee_{i=0}^{n-1}\lambda^{-i}\mathcal{V}(N,\epsilon)$ denotes the coarsest common refinement of the open covers $\lambda^{-i}\mathcal{V}(N,\epsilon)(i = 0, 1, \ldots, L-1)$.

It follows that $\mathcal{N}(\vee_{i=0}^{n-1}\lambda^{-i}\mathcal{V}(N,\epsilon)) \geq L$, where $\mathcal{N}(\mathcal{U})$ implies the minimum among the numbers of elements in subcovers of $\mathcal{U}$. Thus, we have

$$\begin{aligned} h(\lambda) &\geq h(\lambda, \mathcal{V}(N,\epsilon)) \\ &= \lim_{n\to\infty} (1/n) \log \mathcal{N}(\vee_{i=0}^{n-1}\lambda^{-i}\mathcal{V}(N,\epsilon)) \\ &\geq \lim_{n\to\infty} (1/n) \log L \\ &\geq \lim_{n\to\infty} (1/n) \log(C_0/(2\epsilon\lambda^{-n+1})) = \log\lambda. \end{aligned}$$

To complete the proof of Lemma 6, it is sufficient to prove that there exists $n \geq 1$ such that $h(\lambda^n, \mathcal{V}(N,\epsilon)) \leq n \log\lambda$ for any $N > 1$ and $\epsilon > 0$ by Lemma 4. Take any $N > 1$ and $\epsilon > 0$.

For $\omega \in \Omega$, let $\pi_+(\omega)$ be the tile $R \in \mathrm{dom}(\omega)$ such that $C_1/2, 0) \in R$ (see (12)). Let $\pi_-(\omega)$ be the tile $R \in \mathrm{dom}(\omega)$ such that $(x_1 - 0, 0) \in R$, where $[x_1, x_2) \times [y_1, y_2) := \pi_+(\omega)$. Let $[x_1', x_2') \times [y_1', y_2') := \pi_-(\omega)$. Define $\varphi\colon \Omega \to \Pi'$, where

$$\Pi' := ((C_1/2) - 1, C_1/2] \times (\log C_1, 0] \times (\log C_1, 0] \times \mathbb{A} \times \mathbb{A},$$

by

$$\begin{aligned} \varphi(\omega) &= (\varphi_1(\omega), \varphi_2(\omega), \varphi_3(\omega), \varphi_4(\omega), \varphi_5(\omega)) \\ &:= (x_1, y_1, y_1', \omega(\pi_+(\omega)), \omega(\pi_-(\omega))). \end{aligned}$$

Let $\Pi = \varphi(\Omega)$ be the image of φ.

Since the tiles in $\mathrm{dom}(\omega)$ intersecting with $D_{\log(C_1/2)}$ are descendants of $\pi_+(\omega)$ or $\pi_-(\omega)$, by Lemma 1, $\varphi(\omega)$ determines $\omega \in \Omega$ modulo $d_{\log(C_1/2)}$. This means that if the natural distance between $\varphi(\omega)$ and $\varphi(\omega')$ converges to 0, then $d_{\log(C_1/2)}(\omega, \omega') \to 0$.

Define an equivalence relation $\sim_N$ on Ω by $\omega \sim_N \omega'$ if and only if $d_N(\omega, \omega') = 0$. Let $\Omega_N := \Omega/\sim_N$ be the metric space with metric d_N. Take n_0 such that $\lambda^{n_0}C_1/2 \geq e^N$. Then, $\varphi(\lambda^{-n_0}\omega)$ determines ω modulo d_N. Define a mapping $\psi\colon \Pi \to \Omega_N$ by $\psi(x) = \omega$ if and only if $\varphi(\lambda^{-n_0}\omega) = x$. Then, ψ is uniformly continuous.

Take a sufficiently large $n \geq n_0$. For $\omega \in \Omega_N$, consider the following values:

$$(13) \qquad k, i, \sigma \in \{+,-\}, \quad k', i', \sigma' \in \{+,-\}, a, a'$$

such that $\lambda^{-n}\pi_+(\lambda^n\tilde{\omega})$ is the (k,i)-descendent of $\pi_\sigma(\tilde{\omega})$, $\lambda^{-n}\pi_-(\lambda^n\tilde{\omega})$ is the (k',i')-descendent of $\pi_{\sigma'}(\tilde{\omega})$, $a = \varphi_4(\tilde{\omega})$ and $a' = \varphi_5(\tilde{\omega})$, where we put $\tilde{\omega} :=$

$\lambda^{-n_0}\omega$. Let $\mathcal{S}$ be the partition of Ω_N according to the set of these values, so that $\sharp\mathcal{S} \leq C\lambda^n$ holds with C independent of n.

Let $\xi = (\xi_1, \ldots, \xi_5)\colon \Pi \to \Pi$ be the function such that $\psi(\xi(x)) = \lambda^n\psi(x)$ for any $x = (x_1, \ldots, x_5) \in \Pi$. Then for each $S \in \mathcal{S}$, there exist constants d_1, d_2, d_3 such that for any $x \in \psi^{-1}S$,

$$\begin{aligned}\xi_1(x_1, \ldots, x_5) &= \lambda^n(x_1 + d_1 e^{x_{i(\sigma)}}),\\ \xi_2(x_1, \ldots, x_5) &= x_{i(\sigma)} + d_2,\\ \xi_3(x_1, \ldots, x_5) &= x_{i(\sigma')} + d_3,\end{aligned}$$

where σ, σ' are the values in (13) for S, $i(+) = 2$ and $i(-) = 3$.

Since $\lambda^n \circ \psi = \psi \circ \xi$, we have

$$h(\Omega, \lambda^n, \mathcal{V}(N, \epsilon)) \leq h(\Omega_N, \lambda^n) \leq h(\Pi, \xi). \tag{14}$$

On the other hand, $h(\Pi, \xi) \leq n \log \lambda$ holds and the proof of Lemma 6 is completed by (14). The proof of this is not trivial but is omitted, since Lemma 6 will be proved again as $\max_\nu h_\nu(\lambda) = |\log \lambda|$, where ν moves among all λ-invariant probability measures. ∎

LEMMA 7: *Let*

$$\begin{aligned}\Sigma &:= \{\omega \in \Omega; \omega \text{ has a separating line}\},\\ \Sigma_0 &:= \{\omega \in \Omega; y\text{-axis is the separating line of } \omega\}.\end{aligned}$$

Then we have:

(i) *$\Sigma \setminus \Sigma_0$ is dissipative with respect to the G-action. Hence, $\nu(\Sigma \setminus \Sigma_0) = 0$ for any G-invariant probability measure ν on Ω.*

(ii) *For any $\omega \in \Sigma_0$, ω restricted to the right half plane $[0, \infty) \times (-\infty, \infty)$ and to the left half plane $(-\infty, 0) \times (-\infty, \infty)$ are cyclic individually with respect to the G-action. Hence, $\overline{G\omega}$ with respect to the G-action is either cyclic or conjugate to a 2-dimensional irrational rotation with a multiplicative time parameter.*

(iii) *Σ_0 is a finite union of minimal and equicontinuous sets with respect to the G-action. In fact, there is a mapping from the set of pairs $a \in \mathbb{A}$ and i with $0 \leq i < i + 1 < |\sigma(a)|$ onto the set of minimal sets in Σ_0.*

Proof: (i) If $x = u$ is the separating line of $\omega \in \Omega$, then $x = \lambda u$ is the separating line of $\lambda\omega$. Hence, $\Sigma \setminus \Sigma_0$ is dissipative.

(ii) Let $\omega \in \Sigma_0$. Denote by ω^+ the restriction of ω to the right half plane $[0, \infty) \times (-\infty, \infty)$, and by ω^- the restriction of ω to the left half plane $(-\infty, 0) \times$

$(-\infty, \infty)$. Let $(R_i^\pm)_{i\in\mathbb{Z}}$ be the sequence of tiles in $\mathrm{dom}(\omega)$ such that $R_i^\pm$ intersects the line $x = \pm 0$ and $R_{i+1}^\pm$ is a child of $R_i^\pm$ for any $i \in \mathbb{Z}$ ($\pm$ respectively). Let $a_i^\pm := \omega(R_i^\pm)$ ($\pm$ respectively). Define mappings $\sigma_\pm$ from $\mathbb{A}$ to $\mathbb{A}$ by

$$\sigma_+(a) = \sigma(a)_0 \quad \text{and} \quad \sigma_-(a) = \sigma(a)_{|\sigma(a)|-1}.$$

Since $\sigma_\pm(a_i^\pm) = a_{i+1}^\pm (i \in \mathbb{Z})$ ($\pm$ respectively), the sequence $(a_i^\pm)_{i\in\mathbb{Z}}$ is periodic, which also implies that the vertical size of $R_i^\pm$ is also periodic in $i \in \mathbb{Z}$ with the period, say $r^\pm$ which is the minimum period of $(a_i^\pm)_{i\in\mathbb{Z}}$ ($\pm$ respectively). Then,

$$\lambda^+ := \tau^{r^+}(a_0^+)_0^{-1}$$

is the minimum multiplicative cycle of ω^+, while

$$\lambda^- := \tau^{r^-}(a_0^-)^{-1}_{|\sigma^{r^-}(a_0^-)|-1}$$

is the minimum multiplicative cycle of ω^-, that is, $\lambda\omega^\pm = \omega^\pm$ holds for $\lambda = \lambda^\pm$ and $\lambda^\pm$ is the minimum among $\lambda > 1$ with this property ($\pm$ respectively).

Therefore, ω is cyclic with respect to the G-action if λ^+ and λ^- have a common multiplicative base. In this case, the minimum multiplicative cycle of ω is $e^{<\log\lambda^+, \log\lambda^->}$, where $<, >$ implies the least common multiple. Otherwise, the G-action to $\overline{G\omega}$ is conjugate to a 2-dimensional irrational rotation with a multiplicative time parameter.

(iii) We use the notation in the proof of (ii). Take any pair (a, i) with $a \in \mathbb{A}$ and $0 \le i < i + 1 < |\sigma(a)|$. Take any $\omega' \in \Omega$ having a tile $R \in \mathrm{dom}(\omega')$ with $\omega'(R) = a$ such that the y-axis passes between the i-th child of R and the $i+1$-th child of R. Let $\psi(a, i)$ be the set of limit points of $\lambda\omega'$ as $\lambda \in G$ tends to ∞. Note that this does not depend on the choice of ω'. Then, $\psi(a, i)$ is a closed G-invariant subset of Σ_0. Moreover, since the sequence $(\sigma_-^n(\sigma(a)_i), \sigma_+^n(\sigma(a)_{i+1}))_{n=0,1,2,\ldots}$ enters into a cycle after some time, $\psi(a, i)$ is minimal and equicontinuous with respect to the G-action.

To prove that the mapping ψ is onto, take any $\omega \in \Sigma_0$. There exists $\omega_n \in \Omega(\sigma, \tau, g)''$ which converges to ω as $n \to \infty$. We may assume that there exists a pair (a, i) such that for any $n = 1, 2, \ldots$, there exists $R \in \mathrm{dom}(\omega_n)$ with $a = \omega_n(R)$ such that the y-axis passes between the i-th child of R and the $i+1$-th child of R. Then $\omega \in \psi(a, i)$, which proves that ψ is a mapping from the set of pairs (a, i) with $a \in \mathbb{A}$ and $0 \le i < i+1 < |\sigma(a)|$ onto the set of minimal sets in Σ_0 with respect to the G-action. ∎

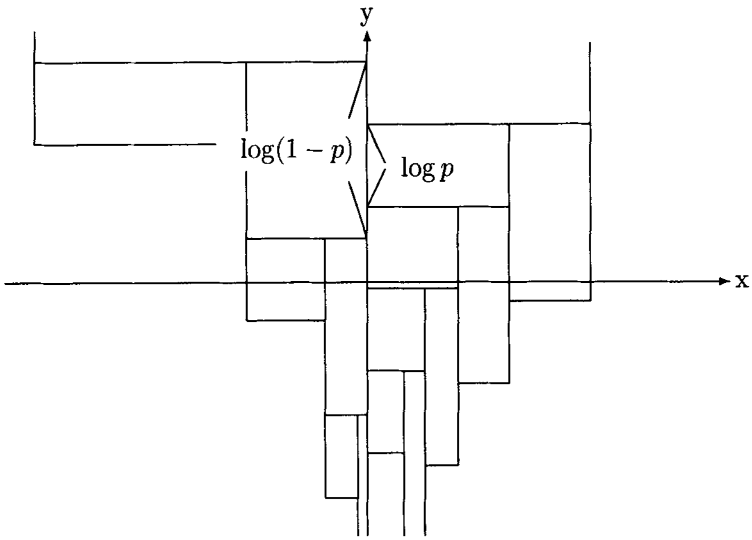

Figure 4. An element in Σ_0 in Example 2

Example 2: Let p with $0 < p < 1$ satisfy that $\log p/\log(1-p)$ is irrational. Let (σ,τ) be a weighted substitution on $\mathbb{A} = \{1\}$ such that $1 \to (1,p)(1,1-p)$. Then, $B(\sigma,\tau) = \mathbb{R}_+$ holds. Let $\Omega = \Omega(\sigma,\tau)$. In this case, elements in Σ_0 are not periodic, but almost periodic as shown in Figure 4. Then, the dynamical system $(\Sigma_0, \lambda(\lambda \in \mathbb{R}_+))$ is isomorphic to $((\mathbb{R}/\mathbb{Z})^2, T_\lambda(\lambda \in \mathbb{R}_+))$ with

$$T_\lambda(x,y) = (x + \log\lambda/\log(1/p), y + \log\lambda/\log(1/(1-p))).$$

LEMMA 8: *Let μ be the unique invariant probability measure on Ω under the additive action. Then, μ is invariant under the G-action satisfying that $h_\mu(\lambda) = |\log\lambda|$ for any $\lambda \in G$. Moreover, if ν is any other G-invariant probability measure on Ω, then $h_\nu(\lambda) < |\log\lambda|$ for any $\lambda \in G$ with $\lambda \neq 1$.*

Proof: The fact that μ is invariant under the G-action is proved in [7]. To prove the lemma, it is sufficient to prove the statements for $\lambda > 1$. Take any G-invariant probability measure ν on Ω which attains the topological entropy of the multiplication by $\lambda_1 \in G$ with $\lambda_1 > 1$, that is, $h_\nu(\lambda_1) = \log\lambda_1$. We assume also that the G-action to Ω is ergodic with respect to ν. Then by Lemma 7, either $\nu(\Sigma_0) = 1$ or $\nu(\Omega \setminus \Sigma) = 1$. In the former case, $h_\nu(\lambda) = 0$ holds for any $\lambda \in G$ since the G-action on Σ_0 is equicontinuous by Lemma 7, which contradicts the assumption that ν attains the topological entropy at $\lambda = \lambda_1 > 1$. Thus, we have $\nu(\Omega \setminus \Sigma) = 1$.

For $\omega \in \Omega$, let $R_0(\omega)$ be such that $(0,0) \in R_0(\omega) \in \mathrm{dom}(\omega)$. Take $a_0 \in \mathbb{A}$ such that

$$\nu(\{\omega \in \Omega; \omega(R_0(\omega)) = a_0\}) > 0.$$

We may assume that $g(a_0) = 1$ (see (5)). Let

$$\begin{aligned}\Omega_1 :=& \{\omega \in \Omega;\ \text{the set } \{\lambda \in G; \lambda\omega(R_0(\lambda\omega)) = a_0\} \\ &\quad \text{is unbounded at 0 and } \infty \text{ simultaneously}\},\\ \Omega_0 :=& \{\omega \in \Omega_1; R_0(\omega) = [x_1, x_2) \times [y_1, y_2) \text{ with } y_1 = 0 \text{ and } \omega(R_0(\omega)) = a_0\}.\end{aligned}$$

For $\omega \in \Omega_0$, let $\lambda_0(\omega)$ be the smallest $\lambda \in G$ with $\lambda > 1$ such that $\lambda\omega \in \Omega_0$. Define a mapping $\Lambda\colon \Omega_0 \to \Omega_0$ by $\Lambda(\omega) := \lambda_0(\omega)\omega$.

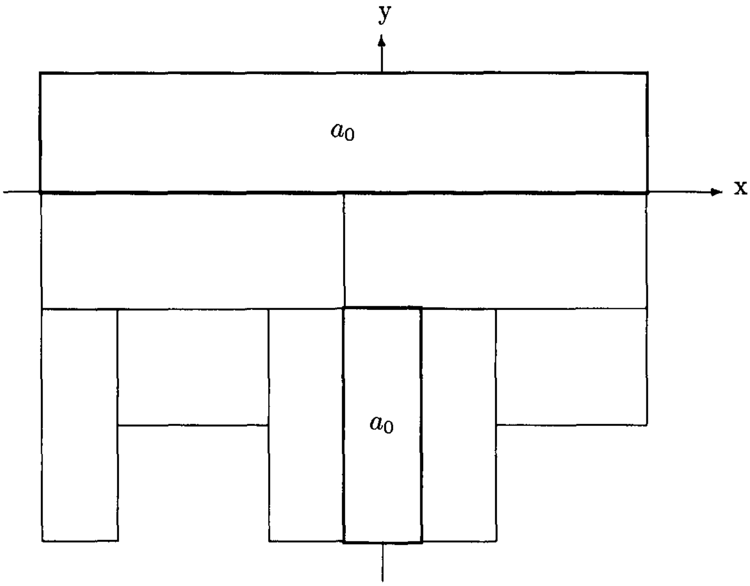

Figure 5. $\omega \in P(2,3)$ with $\lambda_0(\omega) = 8$

For $k = 0, 1, 2, \ldots$ and $i = 0, 1, \ldots, |\sigma^k(a_0)| - 1$, let

$$P(k,i) := \{\omega \in \Omega_0; \lambda_0(\omega)^{-1} R_0(\lambda_0(\omega)\omega) \text{ is the } (k,i)\text{-descendant of } R_0(\omega)\}$$

(see Figure 5) and let

$$\mathcal{P} := \{P(k,i); k = 1, 2, \ldots, 0 \le i < |\sigma^k(a_0)|\}$$

be a measurable partition of Ω_0. Note that $\lambda_0(\omega) = \tau^k(a_0)_i^{-1}$ if $\omega \in P(k,i)$.

Since $\nu(\Omega_1) = 1$ by the ergodicity and

$$\Omega_1 = \bigcup_{P(k,i)\in\mathcal{P}} \bigcup_{\substack{1\le\lambda<\tau^k(a_0)_i^{-1}\\ \lambda\in G}} \lambda P(k,i),$$

there exists a unique Λ-invariant probability measure ν_0 on Ω_0 such that for any Borel set $B \subset \Omega$, we have

$$\nu(B) = C(\nu)^{-1} \sum_{P(k,i)\in\mathcal{P}} \int_1^{\tau^k(a_0)_i^{-1}} \nu_0(\lambda^{-1}B \cap P(k,i))d\lambda/\lambda$$

with

$$C(\nu) := \sum_{P(k,i)\in\mathcal{P}} -\log\tau^k(a_0)_i\nu_0(P(k,i)) < \infty \tag{15}$$

if $G = \mathbb{R}_+$ and

$$\nu(B) = C(\nu)^{-1} \sum_{P(k,i)\in\mathcal{P}} \sum_{\substack{\lambda\in G\\ 1\le\lambda<\tau^k(a_0)_i^{-1}}} \nu_0(\lambda^{-1}B \cap P(k,i))$$

with

$$C(\nu) := \sum_{P(k,i)\in\mathcal{P}} (-\log\tau^k(a_0)_i/\log\beta)\nu_0(P(k,i)) < \infty \tag{16}$$

if $G = \{\beta^n; n \in \mathbb{Z}\}$ with $\beta > 1$.

Since

$$\sum_{P(k,i)\in\mathcal{P}} \tau^k(a_0)_i = 1 \quad \text{and} \quad \sum_{P(k,i)\in\mathcal{P}} \nu_0(P(k,i)) = 1,$$

we have

$$\begin{aligned} H_{\nu_0}(\mathcal{P}) &:= -\sum_{P(k,i)\in\mathcal{P}} \log\nu_0(P(k,i)) \cdot \nu_0(P(k,i)) \\ &\le -\sum_{P(k,i)\in\mathcal{P}} \log\tau^k(a_0)_i \cdot \nu_0(P(k,i)) \end{aligned} \tag{17}$$

by the convexity of $-\log x$. The equality in (17) holds if and only if

$$\nu_0(P(k,i)) = \tau^k(a_0)_i \quad (\forall P(k,i) \in \mathcal{P}). \tag{18}$$

By (15), (16), (17), we have

$$H_{\nu_0}(\mathcal{P}) = -\sum_{P(k,i)\in\mathcal{P}} \log \nu_0(P(k,i)) \cdot \nu_0(P(k,i)) < \infty.$$

For any $\omega, \omega' \in \Omega_0$ such that $\Lambda^k(\omega)$ and $\Lambda^k(\omega')$ belong to the same element in $\mathcal{P}$ for $k = 0, 1, 2, \ldots$, the horizontal position of $R_0(\omega)$, say $[x_1, x_2)$, coincides with that of $R_0(\omega')$. Therefore, ω and ω' restricted to $[x_1, x_2) \times (-\infty, 0]$ coincide. In the same way, if $\Lambda^k(\omega)$ and $\Lambda^k(\omega')$ belong to the same element in $\mathcal{P}$ for any $k \in \mathbb{Z}$, then $R_0 := R_0(\omega) = R_0(\omega')$ holds and all the ancestors of R_0 in ω and ω' coincide as well as their colors. Therefore, ω and ω' restricted to the region covered by the ancestors of R_0 coincide. Hence, if ω or ω' does not have the separating lines, then $\omega = \omega'$ holds.

Since $\nu(\Sigma) = 0$, we have $\nu_0(\Sigma \cap \Omega_0) = 0$. Hence, the above argument implies that $\mathcal{P}$ is a generator of the system (Ω_0, ν, Λ). Thus, $h_{\nu_0}(\Lambda) = h_{\nu_0}(\Lambda, \mathcal{P})$. It follows from (17) that

$$\begin{aligned} h_{\nu_0}(\Lambda) &= h_{\nu_0}(\Lambda, \mathcal{P}) \\ &\le H_{\nu_0}(\mathcal{P}) \\ &\le -\sum_{P(k,i)\in\mathcal{P}} \log \tau^k(a_0)_i \cdot \nu_0(P(k,i)). \end{aligned} \tag{19}$$

The equality in the above that

$$h_{\nu_0}(\Lambda) = -\sum_{P(k,i)\in\mathcal{P}} \log \tau^k(a_0)_i \cdot \nu_0(P(k,i))$$

holds if and only if $(\Lambda^n\mathcal{P})_{n\in\mathbb{Z}}$ is an independent sequence with (18) with respect to ν_0.

Since

$$\begin{aligned} h_\nu(\lambda_1)/\log\lambda_1 &= \frac{h_{\nu_0}(\Lambda)}{\int_{\Omega_0} \lambda_0(\omega) d\nu_0(\omega)} \\ &= \frac{h_{\nu_0}(\Lambda)}{-\sum_{P(k,i)\in\mathcal{P}} \log \tau^k(a_0)_i \cdot \nu_0(P(k,i))}, \end{aligned}$$

$h_\nu(\lambda_1) \le \log\lambda_1$ follows from (19), while the equality holds if and only if $(\Lambda^n\mathcal{P})_{n\in\mathbb{Z}}$ is an independent sequence with (18) with respect to ν_0, that is, $\nu = \mu$ by [6]. which completes the proof of Lemma 8 and Theorem 2. ∎

THEOREM 3: *Let Ω be a numeration system with $G = \mathbb{R}_+$, that is, with the multiplicative $\mathbb{R}_+$-action. Then, the additive action on the probability space Ω with the unique invariant probability measure μ has a pure Lebesgue spectrum.*

Proof: Let $U_t (t \in \mathbb{R})$ and $V_\lambda (\lambda \in \mathbb{R}_+)$ be the groups of the unitary operators on $L^2(\Omega, \mu)$ defined by

$$(U_t f)(\omega) = f(\omega + t), (V_\lambda f)(\omega) = f(\lambda\omega).$$

Let

$$U_t = \int_{-\infty}^{\infty} e^{itu} dE_u (t \in \mathbb{R})$$

be the spectral decomposition of U_t ([3]). Since $U_t V_\lambda = V_\lambda U_{\lambda t}$, we have $dE_u V_\lambda = V_\lambda dE_{\lambda^{-1}u}$.

Take any $f \in L^2(\Omega, \mu)$ with $\int f d\mu = 0$ and $\int |f|^2 d\mu = 1$. Let $m(f)$ be the measure on $\mathbb{R}$ defined by

$$m(f)(S) = \int_S \| dE_u f \|^2$$

for any Borel set $S \subset \mathbb{R}$. Then, $m(f)$ is a probability measure with $m(f)(\{0\}) = 0$. Since $dE_u V_\lambda = V_\lambda dE_{\lambda^{-1}u}$, we have

$$\begin{aligned} m(V_\lambda f)(S) &= \int_S \| dE_u V_\lambda f \|^2 = \int_S \| V_\lambda dE_{\lambda^{-1}u} f \|^2 \\ &= \int_S \| dE_{\lambda^{-1}u} f \|^2 = \int_{\lambda^{-1}S} \| dE_u f \|^2 = m(f)(\lambda^{-1}S). \end{aligned}$$

Moreover, we have

$$\begin{aligned} |(f, V_\lambda f)| &= |\int (dE_u f, dE_u V_\lambda f)| \\ &\le \int \| dE_u f \| \| dE_u V_\lambda f \| = \int \sqrt{dm(f)}\sqrt{dm(V_\lambda f)}. \end{aligned}$$

Since $\lim_{\lambda\to 1} |(f, V_\lambda f)| = 1$, we have

$$\lim_{\lambda\to 1} \int \sqrt{dm(f)}\sqrt{dm(f)\circ\lambda^{-1}} = \lim_{\lambda\to 1} \int \sqrt{dm(f)}\sqrt{dm(V_\lambda f)} = 1.$$

It follows from this that $m(f)$ is absolutely continuous by the following well known argument (see [10], for example).

Suppose to the contrary that $m := m(f)$ is not absolutely continuous. Take a Borel set $S \subset \mathbb{R}$ such that S has Lebesgue measure 0 while $\delta := m(S) > 0$. Denoting $\rho(\lambda) := \int \sqrt{dm}\sqrt{dm\circ\lambda^{-1}}$, we have

$$\begin{aligned} 2(1-\rho(\lambda)) &= \int (\sqrt{dm} - \sqrt{dm\circ\lambda^{-1}})^2 \\ &\ge (\sqrt{m(S)} - \sqrt{m\circ\lambda^{-1}(S)})^2 = (\sqrt{\delta} - \sqrt{m(\lambda^{-1}S)})^2. \end{aligned}$$

Since $2(1-\rho(\lambda)) \to 0$ as $\lambda \to 1$, there exists $\epsilon > 0$ such that for any λ with $1-2\epsilon \le \lambda \le 1+2\epsilon$, $m(\lambda^{-1}S) > \delta/2$ holds. Hence,

$$2\delta\epsilon \le \int_{1-2\epsilon}^{1+2\epsilon} m(\lambda^{-1}S)d\lambda = \int\int_{1-2\epsilon}^{1+2\epsilon} 1_S(\lambda u)d\lambda dm(u).$$

This implies that the set of $u \in \mathbb{R}$ such that $\int_{1-2\epsilon}^{1+2\epsilon} 1_S(\lambda u)d\lambda \ge \delta\epsilon$ has the measure at least $\delta/4$ with respect to m. Since $m(\{0\}) = 0$, this implies that there exists $u \ne 0$ such that $\int_{1-2\epsilon}^{1+2\epsilon} 1_S(\lambda u)d\lambda \ge \delta\epsilon$. Thus, S has Lebesgue measure at least $|u|\delta\epsilon$, which contradicts the assumption that S has Lebesgue measure 0. Thus, $m = m(f)$ is absolutely continuous. ∎

4. The ζ-function

Let $\Omega := \Omega(\sigma, \tau, g)$ satisfy (4) and (5). For $\alpha \in \mathbb{C}$, we define the associated matrices on the suffix set $\mathbb{A} \times \mathbb{A}$ as follows:

$$\begin{aligned} M_\alpha &:= \Big(\sum_{i;\sigma(a)_i = a'} \tau(a)_i^\alpha \Big)_{a,a' \in \mathbb{A}} \\ M_{\alpha,+} &:= (1_{\sigma(a)_0 = a'} \tau(a)_0^\alpha)_{a,a' \in \mathbb{A}} \\ M_{\alpha,-} &:= (1_{\sigma(a)_{|\sigma(a)|-1} = a'} \tau(a)_{|\sigma(a)|-1}^\alpha)_{a,a' \in \mathbb{A}} \end{aligned} \tag{20}$$

Let Θ be the set of **closed orbits** of Ω with respect to the action of G. That is, Θ is the family of subsets ξ of Ω such that $\xi = G\omega$ for some $\omega \in \Omega$ with $\lambda\omega = \omega$ for some $\lambda \in G$ with $\lambda > 1$. We call λ as above a **multiplicative cycle** of ξ. The minimum multiplicative cycle of ξ is denoted by $c(\xi)$. Note that $c(\xi)$ exists, since $\lambda\omega \ne \omega$ for any $\omega \in \Omega$ and $\lambda \in G$ with $1 < \lambda < {C_2}^{-1}$ (see (12)).

We say that $\xi \in \Theta$ has a **separating line** if $\omega \in \xi$ has a separating line. Note that in this case, the separating line is necessarily the y-axis and is in common among $\omega \in \xi$. Denote by Θ_0 the set of $\xi \in \Theta$ with the separating line.

Let

$$L(\sigma) := \{(a,k,i); a \in \mathbb{A}, k = 1,2,\ldots \text{ and } 0 \le i < |\sigma^k(a)| \text{ with } \sigma^k(a)_i = a\}.$$

For (a,k,i) and (a,k',i') in $L(\sigma)$, define the product by

$$(a,k,i)(a,k',i') = (a, k+k', \sum_{j=0}^{i-1} |\sigma^{k'}(\sigma^k(a)_j)| + i').$$

We say that $(a,k,i) \in L(\sigma)$ is **irreducible** if $(a,k,i) = (a,k',i')^h$ does not hold for any $h = 2,3,\ldots$ and $(a,k',i') \in L(\sigma)$.

Let $\xi \in \Theta \setminus \Theta_0$ and $\omega \in \xi$. Then there exists $R = [x_1, x_2) \times [y_1, y_2) \in \mathrm{dom}(\omega)$ with $x_1 < 0 < x_2$. Since $c(\xi)\omega = \omega$, there exists a descendant R' of R such that $\omega(R') = \omega(R) =: a$ and $R = c(\xi)R'$. Let R' be the (k,i)-descendant of R.

LEMMA 9: *For any $\xi \in \Theta \setminus \Theta_0$ with the above setting, the following statements hold.*

(i) $1 \le i < |\sigma^k(a)| - 1$.

(ii) (a, k, i) *is in* $L(\sigma)$ *and irreducible.*

(iii) $c(\xi) = \tau^k(a)_i^{-1}$.

Conversely, any triple $(a, k, i) \in L(\sigma)$ *satisfying* (i), (ii) *determines* $\xi \in \Theta \setminus \Theta_0$ *and* (iii) *follows.*

Proof: (iii) holds since $c(\xi)R' = R$ and R' is the (k,i)-descendant of R.

Since R' is the (k,i)-descendant of R such that $\tau^k(a)_i^{-1}R' = R$, we have

$$\frac{-x_1}{x_2} = \frac{\sum_{j \le i-1} \tau^k(a)_j}{\sum_{j \ge i+1} \tau^k(a)_j}. \tag{21}$$

Since $0 < -x_1/x_2 < \infty$, (i) follows.

If $(a, k, i) = (a, k', i')^\ell$ with $\ell \ge 2$, then the (k', i')-descendant R'' of R also satisfies $\tau^{k'}(a)_{i'}^{-1}R'' = R$ and $\tau^{k'}(a)_{i'}^{-1} = c(\xi)^{1/\ell}$ becomes a cycle of ξ, contradicting the minimality of $c(\xi)$. Thus (ii) follows.

Let us prove the last statement. Assume that $(a, k, i) \in L(\sigma)$ is irreducible satisfying (i). Take any $\omega' \in \Omega$ and $R \in \mathrm{dom}(\omega')$ with $\omega'(R) = a$. Take $t \in \mathbb{R}$ such that $R + t =: [x_1, x_2) \times [y_1, y_2)$ satisfies equation (21). Then $x_1 < 0 < x_2$, and $\tau^k(a)_i^{-n}(\omega' + t)$ converges as $n \to \infty$ to, say, $\omega \in \Omega$ which satisfies $\tau^k(a)_i^{-1}\omega = \omega$. Thus we have $\xi := G\omega \in \Theta \setminus \Theta_0$ which is determined by $(a, k, i) \in L(\sigma)$. (iii) follows by the irreducibility of (a, k, i). ∎

Let $\xi \in \Theta \setminus \Theta_0$ and $\omega \in \xi$. Let R and R' be as in Lemma 9. Let $R_0, R_1, R_2, \ldots$ be the sequence of tiles in ω intersecting the y-axis such that R_{i+1} is a child of $R_i (i = 0, 1, 2, \ldots)$ and $R_0 = R$. Let R_{j+k} be the (k, i_j)-descendant of R_j for any $j = 0, 1, \ldots, k-1$. Then the triple $(\omega(R_j), k, i_j)$ satisfies the conditions (i), (ii) in Lemma 9 and determines ξ for $j = 0, 1, \ldots, k-1$. These k triples are different from each other such that they are all that determine ξ, while k is common among them which we denote by $k(\xi)$.

LEMMA 10: *For $k = 1, 2, \ldots$, we have*

$$tr(M_\alpha^k) - tr(M_{\alpha,+}^k) - tr(M_{\alpha,-}^k) = \sum_{\substack{\xi \in \Theta \setminus \Theta_0 \\ k(\xi) | k}} k(\xi) c(\xi)^{-\frac{k}{k(\xi)}\alpha}. \tag{22}$$

Proof: Note that

$$tr(M_\alpha^k) - tr(M_{\alpha,+}^k) - tr(M_{\alpha,-}^k) = \sum_{\substack{a\in\mathbb{A}, 1\le i<|\sigma^k(a)|-1\\ \sigma^k(a)_i=a}} \tau^k(a)_i^\alpha.$$

For any $\xi \in \Theta \setminus \Theta_0$ with $k(\xi)|k$, there exist exactly $k(\xi)$ number of triples $(a_j, k(\xi), i_j) \in L(\sigma)$ $(j = 0, 1, \dots, k(\xi) - 1)$ satisfying (i), (ii) of Lemma 9 so that $\tau^{k(\xi)}(a_j)_{i_j} = c(\xi)^{-1}$ follows. Therefore,

$$\begin{aligned}\sum_{\substack{a\in\mathbb{A}, 1\le i<|\sigma^k(a)|-1\\ \sigma^k(a)_i=a}} \tau^k(a)_i^\alpha &= \sum_{\substack{(a,k,i)\in L(\sigma)\\ 1\le i<|\sigma^k(a)|-1}} \tau^k(a)_i^\alpha\\ &= \sum_{k'|k} \sum_{\substack{(a,k',i)\in L(\sigma):irreducible\\ 1\le i<|\sigma^{k'}(a)|-1}} \tau^{k'}(a)_i^{(k/k')\alpha}\\ &= \sum_{k'|k} \sum_{\substack{\xi\in\Theta\setminus\Theta_0\\ k(\xi)=k'}} k' c(\xi)^{-(k/k')\alpha}\\ &= \sum_{\substack{\xi\in\Theta\setminus\Theta_0\\ k(\xi)|k}} k(\xi)c(\xi)^{-\frac{k}{k(\xi)}\alpha}. \quad \blacksquare\end{aligned}$$

The following lemma follows from Lemma 7.

LEMMA 11: *The set Θ_0 is a finite set. In fact, we have*

$$\sharp\Theta_0 \le \sum_{a\in\mathbb{A}}(|\sigma(a)| - 1).$$

LEMMA 12: *For $\alpha \in \mathbb{C}$ with $\mathcal{R}(\alpha) > 1$, where $\mathcal{R}(\alpha)$ is the real part of α, we have*

$$\sum_{\xi\in\Theta} |c(\xi)^{-\alpha}| < \infty.$$

Proof: By Lemma 11, it is sufficient to prove that

$$\sum_{\xi\in\Theta\setminus\Theta_0} |c(\xi)^{-\alpha}| < \infty.$$

Since

$$\max_{a\in\mathbb{A}} \sum_{0\le i<|\tau(a)|} \tau(a)_i^{\mathcal{R}(\alpha)} < 1,$$

there exists δ with $0 < \delta < 1$ such that the maximal eigen-value of $M_{\mathcal{R}(\alpha)}$ is less than δ. Hence, by (22) we have

$$\begin{aligned}\sum_{\xi\in\Theta\setminus\Theta_0}|c(\xi)^{-\alpha}| &= \sum_{\xi\in\Theta\setminus\Theta_0} c(\xi)^{-\mathcal{R}(\alpha)}\\ &\leq \sum_{k=1}^{\infty}\sum_{\substack{\xi\in\Theta\setminus\Theta_0\\ k(\xi)\mid k}} k(\xi)c(\xi)^{-\frac{k}{k(\xi)}\mathcal{R}(\alpha)}\\ &\leq \sum_{k=1}^{\infty} tr(M^k_{\mathcal{R}(\alpha)}) \leq \sum_{k=1}^{\infty} C\delta^k < \infty. \qquad \blacksquare\end{aligned}$$

Define the ζ-function of G-action to Ω by

$$\zeta_\Omega(\alpha) := \prod_{\xi\in\Theta}(1-c(\xi)^{-\alpha})^{-1}, \tag{23}$$

where the infinite product converges for any $\alpha \in \mathbb{C}$ with $\mathcal{R}(\alpha) > 1$ by Lemma 12. It is extended to the whole complex plane by the analytic extension.

THEOREM 4: *We have*

$$\zeta_\Omega(\alpha) = \frac{\det(I-M_{\alpha,+})\det(I-M_{\alpha,-})}{\det(I-M_\alpha)}\zeta_{\Sigma_0}(\alpha),$$

where

$$\zeta_{\Sigma_0}(\alpha) := \prod_{\xi\in\Theta_0}(1-c(\xi)^{-\alpha})^{-1}$$

is a finite product with respect to $\xi \in \Theta_0$.

Proof: By the definition of $\zeta_\Omega(\alpha)$ and (22), for any α with $\mathcal{R}(\alpha) > 1$,

$$\begin{aligned}
\zeta_\Omega(\alpha) &= \zeta_{\Sigma_0}(\alpha)\exp\Bigg(\sum_{\xi\in\Theta\setminus\Theta_0} -\log(1-c(\xi)^{-\alpha})\Bigg) \\
&= \zeta_{\Sigma_0}(\alpha)\exp\Bigg(\sum_{\xi\in\Theta\setminus\Theta_0}\sum_{k=1}^{\infty}(1/k)c(\xi)^{-k\alpha}\Bigg) \\
&= \zeta_{\Sigma_0}(\alpha)\exp\Bigg(\sum_{k=1}^{\infty}\sum_{\substack{\xi\in\Theta\setminus\Theta_0 \\ k(\xi)|k}}(k(\xi)/k)c(\xi)^{-\frac{k}{k(\xi)}\alpha}\Bigg) \\
&= \zeta_{\Sigma_0}(\alpha)\exp\Bigg(\sum_{k=1}^{\infty}(1/k)(tr(M_\alpha^k) - tr(M_{\alpha,+}^k) - tr(M_{\alpha,-}^k))\Bigg) \\
&= \zeta_{\Sigma_0}(\alpha)\exp\Bigg(tr\Bigg(\sum_{k=1}^{\infty}(1/k)(M_\alpha^k - M_{\alpha,+}^k - M_{\alpha,-}^k)\Bigg)\Bigg) \\
&= \zeta_{\Sigma_0}(\alpha)\exp(tr(-\log(I-M_\alpha) + \log(I-M_{\alpha,+}) + \log(I-M_{\alpha,-}))) \\
&= \frac{\det(I-M_{\alpha,+})\det(I-M_{\alpha,-})}{\det(I-M_\alpha)}\zeta_{\Sigma_0}(\alpha). \qquad ∎
\end{aligned}$$

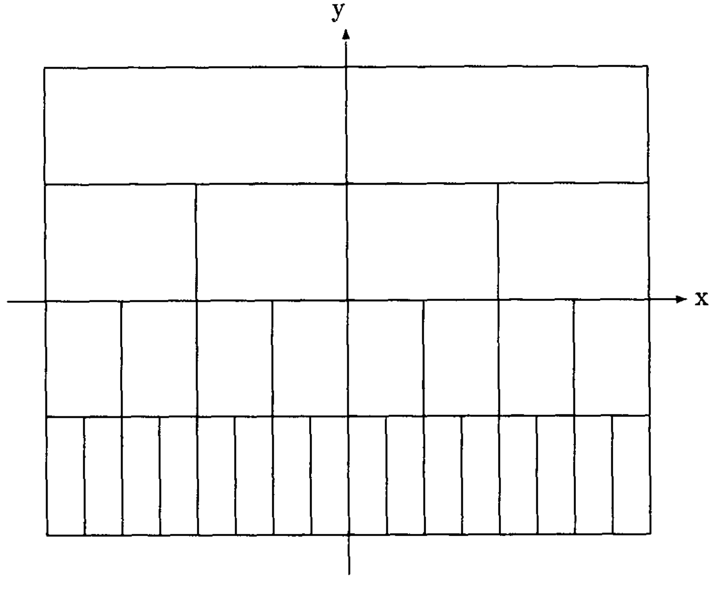

Figure 6. ω_0 in the 2-adic expansion

Example 3: For the 2-adic expansion substitution (σ,τ) defined in Remark 1, $1 \to (1,1/2)(1,1/2)$, define $\Omega := \Omega(\sigma,\tau)$ with $G = \{2^n; n\in\mathbb{Z}\}, g \equiv 1$ (see (5)).

Then we have

$$M_\alpha = 2(1/2)^\alpha, \quad M_{\alpha,+} = M_{\alpha,-} = (1/2)^\alpha.$$

Moreover, we have $\Theta_0 = \{G\omega_0\}$ with ω_0 shown in Figure 6. Since $c(G\omega_0) = 2$ and $\zeta_{\Sigma_0}(\alpha) = (1 - 2^{-\alpha})^{-1}$, we have

$$\begin{aligned}\zeta_\Omega(\alpha) &= \zeta_{\Sigma_0}(\alpha)\frac{(1-(1/2)^\alpha)^2}{1-2(1/2)^\alpha} \\ &= \frac{1-(1/2)^\alpha}{1-2(1/2)^\alpha}.\end{aligned}$$

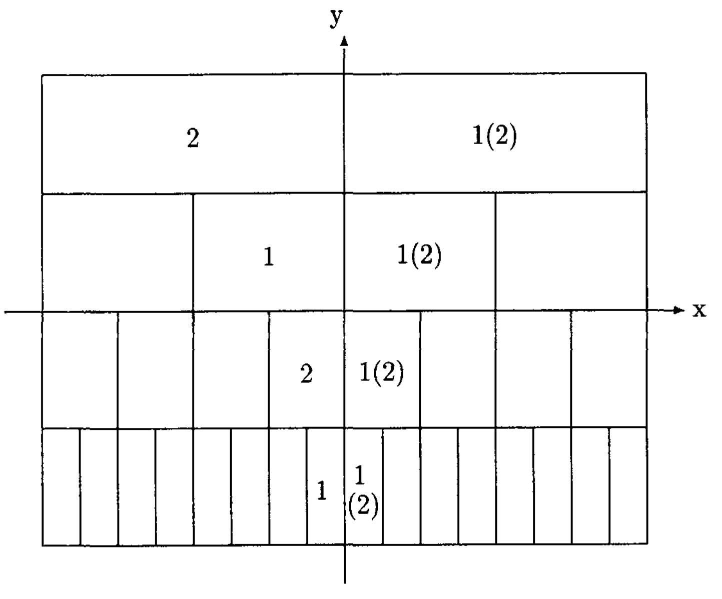

Figure 7. $\omega_1(\omega_2$, respectively) in the Thue–Morse substitution

Example 4: Consider the weighted substitution (σ, τ) in Example 2, that is $1 \to (1,p)(1,1-p)$. Then we have

$$M_\alpha = p^\alpha + (1-p)^\alpha, \quad M_{\alpha,+} = p^\alpha, \quad M_{\alpha,-} = (1-p)^\alpha.$$

Since $\Theta_0 = \emptyset$, we have

$$\zeta_\Omega(\alpha) = \frac{(1-p^\alpha)(1-(1-p)^\alpha)}{1-p^\alpha-(1-p)^\alpha}.$$

Example 5: For the Thue–Morse substitution (σ,τ) defined in Remark 1, $1 \to (1,1/2)(2,1/2), 2 \to (2,1/2)(2,1/2)$, define $\Omega := \Omega(\sigma,\tau)$ with $G = \{2^n; n \in \mathbb{Z}\}$, $g \equiv 1$. Then we have

$$M_\alpha = \begin{pmatrix} (1/2)^\alpha & (1/2)^\alpha \\ (1/2)^\alpha & (1/2)^\alpha \end{pmatrix},$$
$$M_{\alpha,+} = \begin{pmatrix} (1/2)^\alpha & 0 \\ 0 & (1/2)^\alpha \end{pmatrix},$$
$$M_{\alpha,-} = \begin{pmatrix} 0 & (1/2)^\alpha \\ (1/2)^\alpha & 0 \end{pmatrix}.$$

Moreover, we have $\Theta_0 = \{G\omega_1, G\omega_2\}$ with ω_1 and ω_2 shown in Figure 7. Since $c(G\omega_1) = c(G\omega_2) = 4$ and $\zeta_{\Sigma_0}(\alpha) = (1-4^{-\alpha})^{-2}$, we have

$$\begin{aligned}\zeta_\Omega(\alpha) &= \zeta_{\Sigma_0}(\alpha)\frac{(1-(1/2)^\alpha)^2(1-(1/2)^{2\alpha})}{1-2(1/2)^\alpha} \\ &= \frac{1-(1/2)^\alpha}{(1+(1/2)^\alpha)(1-2(1/2)^\alpha)}.\end{aligned}$$

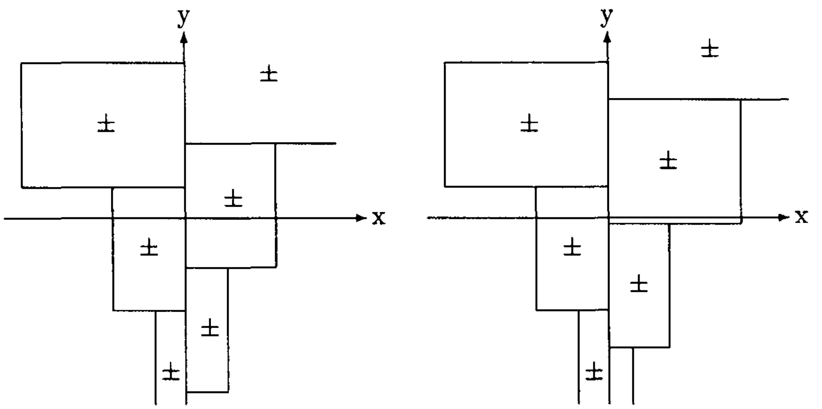

Figure 8. 4 elements in Θ_0 in Example 6 ($\pm$, respectively)

Example 6: For the weighted substitution (σ,τ) in Example 1

$$+ \to (+,4/9)(-,1/9)(+,4/9),$$
$$- \to (-,4/9)(+,1/9)(-,4/9),$$

define $\Omega := \Omega(\sigma,\tau)$, where since $B(\sigma,\tau) = \mathbb{R}$ holds, taking $g \equiv 1$, we have

$$M_\alpha = \begin{pmatrix} 2(4/9)^\alpha & (1/9)^\alpha \\ (1/9)^\alpha & 2(4/9)^\alpha \end{pmatrix},$$
$$M_{\alpha,+} = M_{\alpha,-} = \begin{pmatrix} (4/9)^\alpha & 0 \\ 0 & (4/9)^\alpha \end{pmatrix}.$$

There are 4 elements in Θ_0 determined as in (iii) of Lemma 7 by the pairs $(+,0),(+,1),(-,0),(-,1)$. All of them are different as shown in Figure 8. The $\pm$ corresponds to the $\pm$ in the first coordinate, while $0,1$ corresponds to the vertical gap from the left side tiles to the right side tiles. It is $\pm(\log 9 - 2\log(9/4))$ with $-$ for 0 (left side in Figure 8) and $+$ for 1 (right side in Figure 8). These 4 elements have the same multiplicative cycle $9/4$. Hence, we have

$$\zeta_\Omega(\alpha) = \frac{1}{(1-2(4/9)^\alpha - (1/9)^\alpha)(1-2(4/9)^\alpha + (1/9)^\alpha)}.$$

THEOREM 5: (i) $\zeta_\Omega(\alpha) \neq 0$ *if* $\mathcal{R}(\alpha) \neq 0$.

(ii) *In the region* $\mathcal{R}(\alpha) \neq 0$, α *is a pole of* $\zeta_\Omega(\alpha)$ *with multiplicity* k *if and only if it is a zero of* $\det(I - M_\alpha)$ *with multiplicity* k *for any* $k = 1, 2, \ldots$.

(iii) 1 *is a simple pole of* $\zeta_\Omega(\alpha)$.

Proof: Since $c(\xi) > 1$ for any $\xi \in \Theta$, $1 - c(\xi)^{-\alpha} \neq 0$ if $\mathcal{R}(\alpha) \neq 0$. Hence, $\zeta_{\Sigma_0}(\alpha)$ has neither pole nor zero in the region $\mathcal{R}(\alpha) \neq 0$.

For α with $\mathcal{R}(\alpha) \neq 0$, suppose that $\det(I - M_{\alpha,+}) = 0$, so that $M_{\alpha,+}\xi = \xi$ holds for some nonzero vector $\xi = (\xi_a)_{a\in\mathbb{A}}$. Take $a_0 \in \mathbb{A}$ such that $\xi_{a_0} \neq 0$. Then, for some $k \geq 0$ and $\ell \geq 1$, $\sigma^k(a_0)_0 = \sigma^{k+\ell}(a_0)_0 =: a_1$ holds. Since $\xi_{a_1} = \tau^k(a_0)_0^\alpha \xi_{a_0} \neq 0$ and $\tau^\ell(a_1)_0^\alpha \xi_{a_1} = \xi_{a_1}$, we have $\tau^\ell(a_1)_0^\alpha = 1$. This is impossible since $0 < \tau^\ell(a_1)_0 < 1$.

Thus, $\det(I - M_{\alpha,+})$ has no zero in the region $\mathcal{R}(\alpha) \neq 0$. In the same way, $\det(I - M_{\alpha,-})$ has no zero in the region $\mathcal{R}(\alpha) \neq 0$. These facts with Theorem 4 prove (i), (ii) of Theorem 5.

(iii) Since $M_1{}^t(1,\ldots,1) = {}^t(1,\ldots,1)$, 1 is an eigen-value of M_1, hence is a zero of $\det(I - M_\alpha)$. We prove that it is a simple zero. Let $\mathbb{A} = \{a_1,\ldots,a_r\}$ and $\mathbb{A}' := \mathbb{A} \setminus \{a_1\}$. For a matrix $M = (m_{ij})_{i\in I, j\in J}$ and $I' \subset I, J' \subset J$, let $M[I',J'] := (m_{i,j})_{i\in I', j\in J'}$. Since 1 is the maximum eigen-value of M_1 and σ is primitive, there exists a positive row vector $(\xi_1,\xi_2,\ldots,\xi_r)$ with $\xi_1 = 1$ such that $(\xi_1,\xi_2,\ldots,\xi_r)(I - M_1) = (0,\ldots,0)$. Therefore, since

$$\det(I - M_\alpha) = \det\begin{pmatrix} 1 - \sum_{i=0}^{|\tau(a_1)|-1} \tau(a_1)_i^\alpha & \\ \vdots & (I - M_\alpha)[\mathbb{A},\mathbb{A}']) \\ 1 - \sum_{i=0}^{|\tau(a_r)|-1} \tau(a_r)_i^\alpha & \end{pmatrix},$$

we have

$$\frac{d}{d\alpha}\det(I-M_\alpha)|_{\alpha=1} = \det\begin{pmatrix} \sum_i -\tau(a_1)_i \log\tau(a_1)_i & \\ \vdots & (I-M_1)[\mathbb{A},\mathbb{A}'] \\ \sum_i -\tau(a_r)_i \log\tau(a_r)_i & \end{pmatrix}$$

$$= \det\begin{pmatrix} \sum_{i,j} -\xi_j\tau(a_j)_i \log\tau(a_j)_i & 0\cdots 0 \\ \sum_i -\tau(a_2)_i \log\tau(a_2)_i & \\ \vdots & (I-M_1)[\mathbb{A}',\mathbb{A}'] \\ \sum_i -\tau(a_r)_i \log\tau(a_r)_i & \end{pmatrix}$$

$$= \Big(\sum_{i,j} -\xi_j\tau(a_j)_i \log\tau(a_j)_i\Big)\det((I-M_1)[\mathbb{A}',\mathbb{A}']).$$

We have $\sum_{i,j} -\xi_j\tau(a_j)_i \log\tau(a_j)_i > 0$ and $\det((I-M_1)[\mathbb{A}',\mathbb{A}']) \neq 0$, since the spectral radius of $M_1[\mathbb{A}',\mathbb{A}']$ is strictly less than 1. Hence, $\frac{d}{d\alpha}\det(I-M_\alpha)|_{\alpha=1} \neq 0$ and 1 is a simple zero of $\det(I-M_\alpha)$. By (ii), it is a simple pole of ζ_Ω. ∎

Theorem 6: *For $\Omega = \Omega(\sigma,\eta,g)$, if $B(\sigma,\tau) = \{\lambda^n; n \in \mathbb{Z}\}$ with $\lambda > 1$, then there exist polynomials $p,q \in \mathbb{Z}[z]$ such that $\zeta_\Omega(\alpha) = p(\lambda^\alpha)/q(\lambda^\alpha)$. Conversely, if $\zeta_\Omega(\alpha) = p(\lambda^\alpha)/q(\lambda^\alpha)$ holds for some polynomials $p,q \in \mathbb{Z}[z]$ and $\lambda > 1$, then $B(\sigma,\tau) = \{\lambda^{kn}; n \in \mathbb{Z}\}$ for some positive integer k.*

Proof: Assume that $B(\sigma,\eta) = \{\lambda^n; n \in \mathbb{Z}\}$ with $\lambda > 1$. Let g satisfies (5). Then for any $a \in \mathbb{A}$ and i with $0 \le i < |\sigma(a)|$, there exists $r(a)_i \in \mathbb{Z}$ such that

$$\tau(a)_i = \frac{g(\sigma(a)_i)}{g(a)}\lambda^{r(a)_i}.$$

Hence, we have

$$M_\alpha = \Lambda_\alpha^{-1}\Big(\sum_{\substack{0\le i<|\sigma(a)|\\ \sigma(a)_i=a'}} (\lambda^\alpha)^{r(a)_i}\Big)_{a,a'\in\mathbb{A}}\Lambda_\alpha,$$

$$M_{\alpha,+} = \Lambda_\alpha^{-1}(1_{\sigma(a)_0=a'}(\lambda^\alpha)^{r(a)_0})_{a,a'\in\mathbb{A}}\Lambda_\alpha,$$

$$M_{\alpha,-} = \Lambda_\alpha^{-1}(1_{\sigma(a)_{|\sigma(a)|-1}=a'}(\lambda^\alpha)^{r(a)_{|\sigma(a)|-1}})_{a,a'\in\mathbb{A}}\Lambda_\alpha,$$

where $\Lambda_\alpha = (g(a)^\alpha 1_{a'=a})_{a,a'\in\mathbb{A}}$ is a diagonal matrix. Therefore, $\det(I-M_\alpha)$, $\det(I-M_{\alpha,+})$ and $\det(I-M_{\alpha,-})$ are polynomials in λ^α divided possibly by $(\lambda^\alpha)^n$ for some positive integer n. Since $c(\xi) = \lambda^n$ for some positive integer n for any $\xi \in \Theta_0$ and Θ_0 is a finite set, $\zeta_{\Sigma_0}(\alpha)^{-1}$ is a polynomial in $\lambda^{-\alpha}$. Thus, $\zeta_\Omega(\alpha) = p(\lambda^\alpha)/q(\lambda^\alpha)$ for some polynomials $p,q \in \mathbb{Z}[z]$.

Conversely, assume that $\zeta_\Omega(\alpha) = p(\lambda^\alpha)/q(\lambda^\alpha)$ for some polynomials $p, q \in \mathbb{Z}[z]$. Then we have

$$\Pi_{\xi\in\Theta}(1 - c(\xi)^{-\alpha}) = q(\lambda^\alpha)/p(\lambda^\alpha)$$

on $\mathcal{R}(\alpha) > 1$. Comparing their expansions as Dirichlet series in α, we have that $c(\xi) \in \{\lambda^n; n \in \mathbb{Z}\}$ for any $\xi \in \Theta$ ([5]). Therefore, $B(\sigma,\tau) = \{c(\xi); \xi \in \Sigma\} \subset \{\lambda^n; n \in \mathbb{Z}\}$. Thus, $B(\sigma,\tau) = \{\lambda^{kn}; n \in \mathbb{Z}\}$ for some positive integer k. ∎

THEOREM 7: *If $B(\sigma,\tau) = \{\lambda^n; n \in \mathbb{Z}\}$, then λ is an algebraic number.*

Proof: Let $\Omega := \Omega(\sigma,\tau,g)$. By Theorem 6, there exist polynomials $p, q \in \mathbb{Z}[z]$ such that $\zeta_\Omega(\alpha) = p(\lambda^\alpha)/q(\lambda^\alpha)$. By Theorem 5, 1 is a pole of $\zeta_\Omega(\alpha)$. Hence $q(\lambda) = 0$, which implies that λ is algebraic. ∎

5. The β-expansion system

Let β be an algebraic integer with $\beta > 1$ such that 1 has the following periodic β-expansion:

$$\begin{aligned} &1 = (b_1 0^{i_1-1} b_2 0^{i_2-1} \cdots b_k 0^{i_k-1})^\infty, \\ &b_1, b_2, \ldots, b_k \in \{1, 2, \ldots, \lfloor\beta\rfloor\}, \\ &i_1, i_2, \ldots, i_k \in \{1, 2, \ldots\}, \end{aligned}$$

where $()^\infty$ implies the infinite time repetition of $()$. Let $n := i_1 + i_2 + \cdots + i_k \geq 1$ and assume that n is the minimum period of the above sequence. Since the above sequence is the expansion of 1, we have the solution of the following equation in $a_1, a_2, \ldots, a_{k+1}$ with $a_1 = a_{k+1} = 1$ and $0 < a_j < 1$ $(j = 2, \ldots, k)$:

$$a_j = b_j\beta^{-1} + a_{j+1}\beta^{-i_j} \, (j = 1, 2, \ldots, k).$$

Let $\mathbb{A} := \{1, 2, \ldots, k\}$ and define a weighted substitution (σ,τ) by

$$\begin{gathered} j \to (1, (1/a_j)\beta^{-1})^{b_j}(j+1, (a_{j+1}/a_j)\beta^{-i_j}) \\ (j = 1, 2, \ldots, k-1), \\ k \to (1, (1/a_k)\beta^{-1})^{b_k}(1, (a_{k+1}/a_k)\beta^{-i_k}), \end{gathered}$$

where $(,)^k$ implies the k-time repetition of $(,)$. Then σ is primitive and $B(\sigma,\tau) = \{\beta^n; n \in \mathbb{Z}\}$. Define $g\colon \mathbb{A} \to \mathbb{R}_+$ by $g(j) := a_j$. Then g satisfies (5) and $\Omega(\sigma,\tau,g)$ is a numeration system by Theorem 2. We denote $\Omega(\beta) := \Omega(\sigma,\tau,g)$ and $\Omega(\beta)$ is called the **β-expansion system.**

The β-expansion system has been studied by many authors, for example, S. Ito and Y. Takahashi ([6]) where the ζ-function is obtained. Here, we give the formula again as a corollary of Theorem 4. There is little difference between them. In [6], the ζ-function is for the right-half space $\{\omega^+; \omega \in \Omega\}$, while ours is for the full space Ω, so that ours is multiplied by the former by $(1-\beta^{\alpha})/(1-\beta^{-n\alpha})$.

THEOREM 8: *We have*

$$\zeta_{\Omega(\beta)}(\alpha) = \frac{1-\beta^{-\alpha}}{1-\sum_{j=1}^{k} b_j \beta^{-(i_1+\cdots+i_{j-1}+1)\alpha} - \beta^{-n\alpha}}.$$

Proof: We have

$$\det(I - M_\alpha) =$$
$$\begin{vmatrix} 1-b_1(\frac{1}{a_1\beta})^\alpha & -(\frac{a_2}{a_1\beta^{i_1}})^\alpha & 0 & 0 & \cdots & 0 & 0 \\ -b_2(\frac{1}{a_2\beta})^\alpha & 1 & \ddots & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ -b_{k-1}(\frac{1}{a_{k-1}\beta})^\alpha & 0 & 0 & 0 & \cdots & 1 & -(\frac{a_k}{a_{k-1}\beta^{i_{k-1}}})^\alpha \\ -b_k(\frac{1}{a_k\beta})^\alpha - (\frac{a_{k+1}}{a_k\beta^{i_k}})^\alpha & 0 & 0 & 0 & \cdots & 0 & 1 \end{vmatrix}$$
$$= 1 - \sum_{j=1}^{k} b_j \beta^{-(i_1+\cdots+i_{j-1}+1)\alpha} - \beta^{-n\alpha},$$

$$\det(I - M_{\alpha,+}) = \begin{vmatrix} 1-(\frac{1}{a_1\beta})^\alpha & 0 & \cdots & \cdots & 0 \\ -(\frac{1}{a_2\beta})^\alpha & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -(\frac{1}{a_{k-1}\beta})^\alpha & 0 & \cdots & 1 & 0 \\ -(\frac{1}{a_k\beta})^\alpha & 0 & \cdots & 0 & 1 \end{vmatrix} = 1 - \beta^{-\alpha}$$

and

$$\det(I - M_{\alpha,-}) =$$
$$\begin{vmatrix} 1 & -(\frac{a_2}{a_1\beta^{i_1}})^\alpha & 0 & \cdots & 0 \\ 0 & 1 & -(\frac{a_3}{a_2\beta})^\alpha & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & -(\frac{a_k}{a_{k-1}\beta^{i_{k-1}}})^\alpha \\ -(\frac{a_{k+1}}{a_k\beta^{i_k}})^\alpha & 0 & \cdots & 0 & 1 \end{vmatrix}$$
$$= 1 - \beta^{-n\alpha}.$$

Let $\xi \in \Theta_0$ and $\omega \in \xi$. Let $R_i^{\pm} (i \in \mathbb{Z})$ be the sequence of tiles in $\mathrm{dom}(\omega)$ from up to down intersecting with the line $y = \pm 0$ and $(0, \pm 0) \in R_0^{\pm}$ ($\pm$ respectively). Then, we have

$$\omega(R_{-1}^+)\omega(R_0^+)\omega(R_1^+)\omega(R_2^+) = \cdots\cdots 11111 \cdots\cdots$$
$$\cdots \omega(R_{-1}^-)\omega(R_0^-)\omega(R_1^-)\omega(R_2^-) \cdots = \cdots 12 \cdots k 12 \cdots k \cdots.$$

Hence, the minimum multiplicative cycle of ω on the right half plane is β, while on the left half plane it is β^n. Thus, $c(\xi) = \beta^n$. Moreover, the upper half tiling and the lower half tiling are synchronized, so that the horizontal position of any tile R_i^- with $\omega(R_i^-) = 1$ coincides with that of R_j^+ for some j. Therefore, ξ is the unique element in Θ_0. Hence, $\zeta_{\Sigma_0}(\alpha) = 1 - \beta^{-n\alpha}$.

Combining these results using Theorem 4, we have

$$\zeta_{\Omega(\beta)}(\alpha) = \frac{1 - \beta^{-\alpha}}{1 - \sum_{j=1}^{k} b_j \beta^{-(i_1 + \cdots + i_{j-1} + 1)\alpha} - \beta^{-n\alpha}}. \qquad ∎$$

Example 7: Let $\beta = (1 + \sqrt{5})/2$ be the golden number. Then, the expansion of 1 is $(10)^\infty$. Therefore, $\mathbb{A} = \{1\}$ and (σ, τ) is

$$1 \to (1, \beta^{-1})(1, \beta^{-2}).$$

By Theorem 8, we have

$$\zeta_{\Omega(\beta)}(\alpha) = \frac{1 - \beta^{-\alpha}}{1 - \beta^{-\alpha} - \beta^{-2\alpha}}.$$

Example 8: Let us consider the β-expansion system with $\beta > 1$ such that $\beta^3 - \beta^2 - \beta - 1 = 0$. Then the expansion of 1 is $(110)^\infty$ and the corresponding weighted substitution is

$$1 \to (1, \beta^{-1})(2, \beta^{-2} + \beta^{-3}),$$
$$2 \to \left(1, \frac{\beta}{\beta + 1}\right)\left(1, \frac{1}{\beta + 1}\right).$$

By Theorem 8, we have

$$\zeta_{\Omega(\beta)}(\alpha) = \frac{1 - \beta^{-\alpha}}{1 - \beta^{-\alpha} - \beta^{-2\alpha} - \beta^{-3\alpha}}.$$

We will discuss this example in the next section.

6. Homogeneous cocycles and fractals

Let $\Omega := \Omega(\sigma, \tau, g)$ satisfy (4) and (5).

A continuous function $F: \Omega \times \mathbb{R} \to \mathbb{C}$ is called a **cocycle** on Ω if

$$F(\omega, t+s) = F(\omega, t) + F(\omega + t, s) \tag{24}$$

holds for any $\omega \in \Omega$ and $s, t \in \mathbb{R}$. A cocycle F on Ω is called **α-homogeneous** if

$$F(\lambda\omega, \lambda t) = \lambda^\alpha F(\omega, t)$$

for any $\omega \in \Omega$, $\lambda \in G$ and $t \in \mathbb{R}$, where α is a given complex number. A cocycle $F(\omega, t)$ on Ω is called **adapted** if there exists a function Ξ: $\mathbb{A} \times \mathbb{R}_+ \to \mathbb{C}$ such that

$$F(\omega, x_2) - F(\omega, x_1) = \Xi(\omega(R), x_2 - x_1) \tag{25}$$

for any $\omega \in \Omega$ and tile $R := [x_1, x_2) \times [y_1, y_2) \in \mathrm{dom}(\omega)$.

In [7], nonzero adapted α-homogeneous cocycles on Ω with $0 < \alpha < 1$ are characterized. In fact, we have

THEOREM 9: *A nonzero adapted α-homogeneous cocycle on Ω is characterized by (25) with α and Ξ satisfying $\mathcal{R}(\alpha) > 0$ and there exists a nonzero vector $\xi = (\xi_a)_{a\in\mathbb{A}}$ such that $M_\alpha \xi = \xi$ (see (20)) and $\Xi(\omega(R), x_2 - x_1) = (x_2 - x_1)^\alpha \xi_{\omega(R)}$ for any tile $R := [x_1, x_2) \times [y_1, y_2) \in \mathrm{dom}(\omega)$. Hence, a nonzero adapted α-homogeneous cocycle exists if and only if $\mathcal{R}(\alpha) > 0$ and α is a pole of $\zeta_\Omega(\alpha)$.*

Proof: The last part of the theorem follows from Theorem 5. The condition $\mathcal{R}(\alpha) > 0$ is necessary for the convergence of $F(\omega, t)$ with (25) for a general $t \in \mathbb{R}$. ∎

It is known [7] that

THEOREM 10: *Let μ be the unique invariant probability measure on Ω under the additive action. Let $0 < \alpha < 1$. For a nonzero α-homogeneous cocycle F on Ω, we have the following results.*

(i) *There exists a constant C such that*

$$|F(\omega, t) - F(\omega, s)| \leq C|t - s|^\alpha$$

for any $\omega \in \Omega$ and $s, t \in \mathbb{R}$. That is, the functions $F(\omega, t)$ on t for $\omega \in \Omega$ are uniformly α-Hölder continuous.

(ii) *For any* $\omega \in \Omega$ *and* $t \in \mathbb{R}$,

$$\limsup_{s \downarrow 0} \frac{1}{s^{\alpha}} |F(\omega, t+s) - F(\omega, t)| > 0$$

holds. That is, for any $\omega \in \Omega$ *the function* $F(\omega, \cdot)$ *is nowhere locally* α'*-Hölder continuous for any* $\alpha' > \alpha$. *In particular,* $F(\omega, \cdot)$ *is nowhere differentiable.*

(iii) *The stochastic process* $F(\omega, t)$ *with time parameter* $t \in \mathbb{R}$ *and random element* $\omega \in \Omega$ *with respect to* μ *has a strictly ergodic stationary increment having 0-entropy.*

(iv) $F(\omega, \lambda t)$ *has the same law as* $\lambda^{\alpha} F(\omega, t)$ *for any* $\lambda \in G$. *Hence, the process* $F(\omega, t)$ *is* α*-self similar if* $G = \mathbb{R}_{+}$.

(v) $\int F(\omega, t) d\mu(\omega) = 0$ *for any* $t \in \mathbb{R}$.

Example 9: Take Ω in Examples 1 and 6. Since

$$M_{1/2} = \begin{pmatrix} 4/3 & 1/3 \\ 1/3 & 4/3 \end{pmatrix},$$

$\xi = \binom{1}{-1}$ is an eigenvector of $M_{1/2}$ with eigen-value 1. Let F be the 1/2-homogeneous adapted cocycle on Ω defined by the equation:

$$F(\omega, x_2) - F(\omega, x_1) = \pm (x_2 - x_1)^{1/2}$$

if there exists a tile $[x_1, x_2) \times [y_1, y_2)$ in ω with color $\pm$, respectively (see Theorem 9).

Then, $F(\omega, t)$ is a 1/2-self-similar process with respect to the unique invariant measure μ under the additive action, called N-process, which is discussed in the next section.

Consider the family of functions $(F(\omega + n, 1))_{n=0,1,2,\ldots}$ on the probability space (Ω, μ). Since

$$\begin{aligned} &nE[F(\omega,1)^2] = E[(n^{1/2}F(\omega,1))^2] = E[F(n\omega,n)^2] = E[F(\omega,n)^2] \\ &= E[F(\omega,1)^2] + E[(F(\omega,n) - F(\omega,1))^2] + 2E[F(\omega,1)(F(\omega,n) - F(\omega,1))] \\ &= E[F(\omega,1)^2] + E[F(\omega+1,n-1)^2] + 2E[F(\omega,1)(F(\omega,n) - F(\omega,1))] \\ &= E[F(\omega,1)^2] + (n-1)E[F(\omega,1)^2] + 2E[F(\omega,1)(F(\omega,n) - F(\omega,1))], \end{aligned}$$

we have

$$E[F(\omega,1)(F(\omega,n) - F(\omega,1))] = 0$$

for any $n = 1, 2, \ldots$. Therefore,

$$\begin{aligned} E[F(\omega,1)(F(\omega+n,1)] =& E[F(\omega,1)(F(\omega,n+1)-F(\omega,n))] \\ =& E[F(\omega,1)(F(\omega,n+1)-F(\omega,1))] \\ & - E[F(\omega,1)(F(\omega,n)-F(\omega,1))] \\ =& 0 \end{aligned}$$

for any $n = 1, 2, \ldots$. Hence, for any $n < m$,

$$E[F(\omega+n,1)(F(\omega+m,1)] = E[F(\omega,1)(F(\omega+m-n,1)] = 0.$$

This implies that the family of functions $(F(\omega+n,1))_{n=0,1,\ldots}$ is noncorrelated.

Let $\mathcal{I}(\Omega)$ be the set of $\omega \in \Omega$ such that there exists $[x_1,x_2)\times[y_1,y_2) \in \mathrm{dom}(\omega)$ satisfying $x_1 = 0$ and $y_1 \le 0 < y_2$. An element $\omega \in \mathcal{I}(\Omega)$ is called an **integer** in Ω. Let

$$\mathcal{II}(\Omega) := \{(\omega,t) \in \mathcal{I}(\Omega) \times \mathbb{R}; \omega + t \in \mathcal{I}(\Omega)\}.$$

A continuous function F: $\mathcal{II}(\Omega) \to \mathbb{C}$ is called a cocycle on $\mathcal{I}(\Omega)$ if (24) is satisfied for any $\omega \in \mathcal{I}(\Omega)$ and $t, s \in \mathbb{R}$ such that $(\omega,t) \in \mathcal{II}(\Omega)$ and $(\omega, t+s) \in \mathcal{II}(\Omega)$.

A cocycle F on $\mathcal{I}(\Omega)$ is called **adapted** if there exists a function Ξ: $\mathbb{A}\times\mathbb{R}_+ \to \mathbb{C}$ such that (25) is satisfied for any $\omega \in \mathcal{I}(\Omega)$ and tile $[x_1,x_2)\times[y_1,y_2) \in \mathrm{dom}(\omega)$ with $y_2 > 0$. Let $\alpha \in \mathbb{C}$. A cocycle F on $\mathcal{I}(\Omega)$ is called **α-homogeneous** if

$$F(\lambda\omega, \lambda t) = \lambda^\alpha F(\omega,t)$$

for any $(\omega,t) \in \mathcal{II}(\Omega)$ and $\lambda \in G$ with $(\lambda\omega,\lambda t) \in \mathcal{II}(\Omega)$. Note that if $(\omega,t) \in \mathcal{II}(\Omega)$, then for any $\lambda \in G$ with $\lambda > 1$, $(\lambda\omega,\lambda t) \in \mathcal{II}(\Omega)$ holds.

A cocycle F on $\mathcal{I}(\Omega)$ is called a **coboundary** on $\mathcal{I}(\Omega)$ if there exists a continuous function G: $\mathcal{I}(\Omega) \to \mathbb{R}^k$ such that

$$F(\omega,t) = G(\omega+t) - G(\omega)$$

for any $(\omega,t) \in \mathcal{II}(\Omega)$.

The following theorem is proved in [4].

THEOREM 11: *A nonzero adapted α-homogeneous cocycle on $\mathcal{I}(\Omega)$ with $\mathcal{R}(\alpha) < 0$ is characterized by (25) with Ξ satisfying that there exists a nonzero vector $\xi = (\xi_a)_{a\in\mathbb{A}}$ such that $M_\alpha\xi = \xi$ (see (20)) and $\Xi(\omega(R), x_2-x_1) = (x_2-x_1)^\alpha \xi_{\omega(R)}$ for any tile $R := [x_1,x_2)\times[y_1,y_2) \in \mathrm{dom}(\omega)$ with $y_2 > 0$. Hence, a nonzero adapted α-homogeneous cocycle on $\mathcal{I}(\Omega)$ with $\mathcal{R}(\alpha) < 0$ exists if and only if α is a pole of $\zeta_\Omega(\alpha)$. Moreover, any cocycle like this is a coboundary.*

Example 10: Let us consider the β-expansion system in Example 8. Denote $\Omega := \Omega(\beta)$. The associated matrix is

$$M_\alpha = \begin{pmatrix} \beta^{-\alpha} & (\beta^{-2}+\beta^{-3})^\alpha \\ \frac{\beta^\alpha+1}{(\beta+1)^\alpha} & 0 \end{pmatrix}.$$

Let γ be one of the complex solutions of the equation $z^3 - z^2 - z - 1 = 0$. Then $|\gamma| < 1$. Let $\alpha \in \mathbb{C}$ be such that $\gamma = \beta^\alpha$. Then $\mathcal{R}(\alpha) < 0$. Since we have

$$M_\alpha \begin{pmatrix} 1 \\ \delta \end{pmatrix} = \begin{pmatrix} 1 \\ \delta \end{pmatrix}$$

with

$$\delta := \frac{\beta^\alpha + 1}{(\beta+1)^\alpha},$$

there exists an α-homogeneous adapted cocycle F on $\mathcal{I}(\Omega)$ satisfying

$$F(\omega, x_2) - F(\omega, x_1) = \begin{cases} (x_2 - x_1)^\alpha & (\omega(R) = 1) \\ \delta(x_2 - x_1)^\alpha & (\omega(R) = 2) \end{cases}$$

if there exists $R := [x_1, x_2) \times [y_1, y_2) \in \mathrm{dom}(\omega)$ with $y_2 > 0$.

For $\omega \in \mathcal{I}(\Omega)$, let $R_0(\omega)$ be the tile $[x_0, \tilde{x}_0) \times [y_0, \tilde{y}_0) \in \mathrm{dom}(\omega)$ such that $x_0 = 0$ and $y_0 \le 0 < \tilde{y}_0$. For $i = 0, 1, 2, \ldots$, let R_i be the i-th ancestor of $R_0(\omega)$. Let $R_i = [x_i, \tilde{x}_i) \times [y_i, \tilde{y}_i)$. Let

$$G(\omega) := \sum_{i=0}^{\infty} (x_i - x_{i+1})^\alpha.$$

Since, if $x_i > x_{i+1}$, then there exists a tile $[x_{i+1}, x_i) \times [y_{i+1}, y_{i+1} + \log\beta)$ with color 1 in ω, we have

$$F(\omega, x_i) - F(\omega, x_{i+1}) = (x_i - x_{i+1})^\alpha$$

for any $i = 0, 1, \ldots$.

Take any $t \in \mathbb{R}$ such that $(\omega, t) \in \mathcal{II}(\Omega)$. Let $(R'_i)_{i=1,2,\ldots}$ and $(x'_i)_{i=0,1,\ldots}$ be the sequences as above for $\omega + t$ instead of ω. Then there exist $i_0 \ge 1, j_0 \ge 1$ such that $R'_{i_0+k} = R_{j_0+k} + t$ for any $k = 0, 1, \ldots$. Then, since $x'_{j_0+k} = x_{i_0+k} - t$ for any $k = 0, 1, \ldots$, we have

$$\begin{aligned} G(\omega + t) - G(\omega) &= \sum_{i=0}^{j_0-1} (x'_i - x'_{i+1})^\alpha - \sum_{i=0}^{i_0-1} (x_i - x_{i+1})^\alpha \\ &= -F(\omega + t, x'_{j_0}) + F(\omega, x_{i_0}) = F(\omega, t). \end{aligned}$$

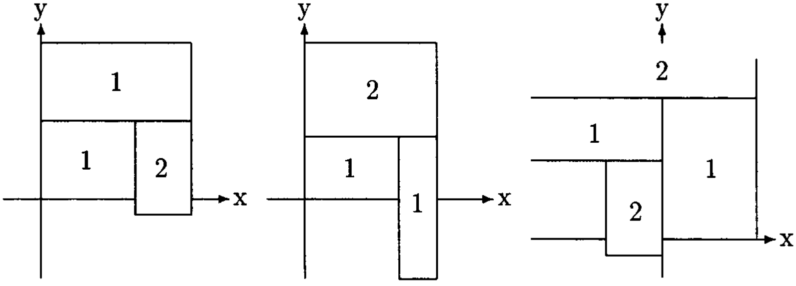

Figure 9. $\beta\mathcal{I}(\Omega)^1, \beta\mathcal{I}(\Omega)^2, \beta^2\mathcal{I}(\Omega)^2 + \beta$

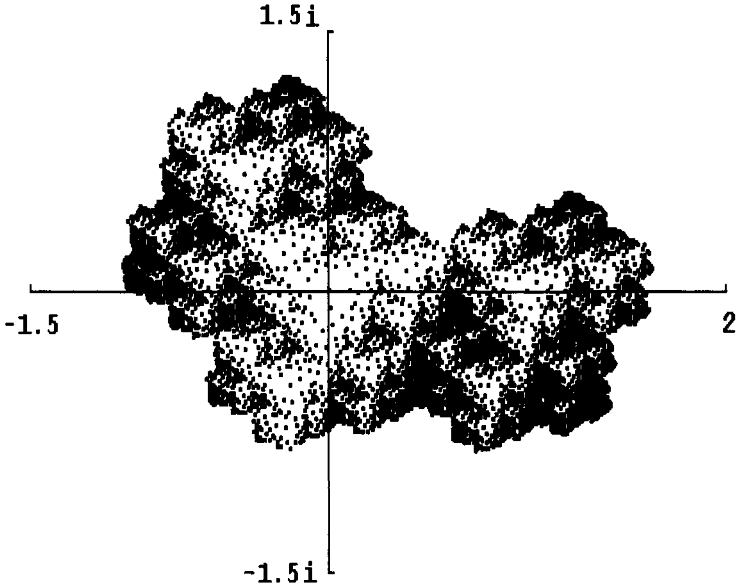

Figure 10. $G(\mathcal{I}(\Omega))$

Thus the α-homogeneous cocycle F is a coboundary with coboundary function G. The set $G(\mathcal{I}(\Omega))$ is known as a Rauzy fractal. For $\omega \in \mathcal{I}(\Omega)$, let

$$\mathcal{I}(\Omega)^i = \{\omega \in \mathcal{I}(\Omega); \omega(R_0(\omega)) = i\} \quad (i = 1, 2)$$

and

$$G_i = G(\mathcal{I}(\Omega)^i) \quad (i = 1, 2).$$

Considering the children of the tile $R_0(\omega)$, we have a set equation

$$\begin{aligned}\mathcal{I}(\Omega)^1 &= \beta\mathcal{I}(\Omega)^1 \cup \beta\mathcal{I}(\Omega)^2 \cup (\beta^2\mathcal{I}(\Omega)^2 + \beta),\\ \mathcal{I}(\Omega)^2 &= \beta\mathcal{I}(\Omega)^1 + 1.\end{aligned}$$

Hence, the following set equation holds:

$$\begin{aligned}G_1 &= \gamma G_1 \cup \gamma G_2 \cup (\gamma^2 G_2 + \gamma),\\ G_2 &= \gamma G_1 + 1.\end{aligned}$$

$G_1 \cup G_2$ is shown in Figure 10.

Remark 2: The above Rauzy fractal is usually introduced by the substitution $1 \to 12$, $2 \to 13$, $3 \to 1$ (S. Ito and P. Arnoux [1]). We modify it canonically to the weighted substitution in Example 8 (see Remark 1). The set equation is usually denoted as

$$\begin{aligned}G_1 &= \gamma G_1 \cup \gamma G_2 \cup \gamma G_3,\\ G_2 &= \gamma G_1 + 1,\\ G_3 &= \gamma G_2 + 1,\end{aligned}$$

which is equivalent to ours.

7. The N-process

We consider the N**-process** defined in Example 9. It is defined as a $(1/2)$-homogeneous cocycle F on the space $\Omega = \Omega(\sigma, \tau)$ with the weighted substitution (σ, τ) in Example 1. Hence, F is defined by

$$F(\omega, x_2) - F(\omega, x_1) = \pm(x_2 - x_1)^{1/2} \tag{26}$$

if there is a tile $[x_1, x_2) \times [y_1, y_2) \in \mathrm{dom}(\omega)$, where $\pm$ corresponds to the color of the tile.

Take $\omega_0 \in \Omega$ which has a tile $R_0 := [0, 1) \times [0, \log(4/9))$ with $\omega_0(R_0) = +$. Then $F(\omega_0, 1) = F(\omega_0, 1) - F(\omega_0, 0) = 1$ by (26). Since R_0 has 3 children $R_{1,0}, R_{1,1}, R_{1,2}$ with colors $+, -, +$ and the vertical sizes $4/9, 1/9, 4/9$, we have $F(\omega_0, 4/9) = 2/3$ and $F(\omega_0, 5/9) - F(\omega_0, 4/9) = -1/3$ by (26). Hence

$F(\omega_0, 5/9) = 1/3$.

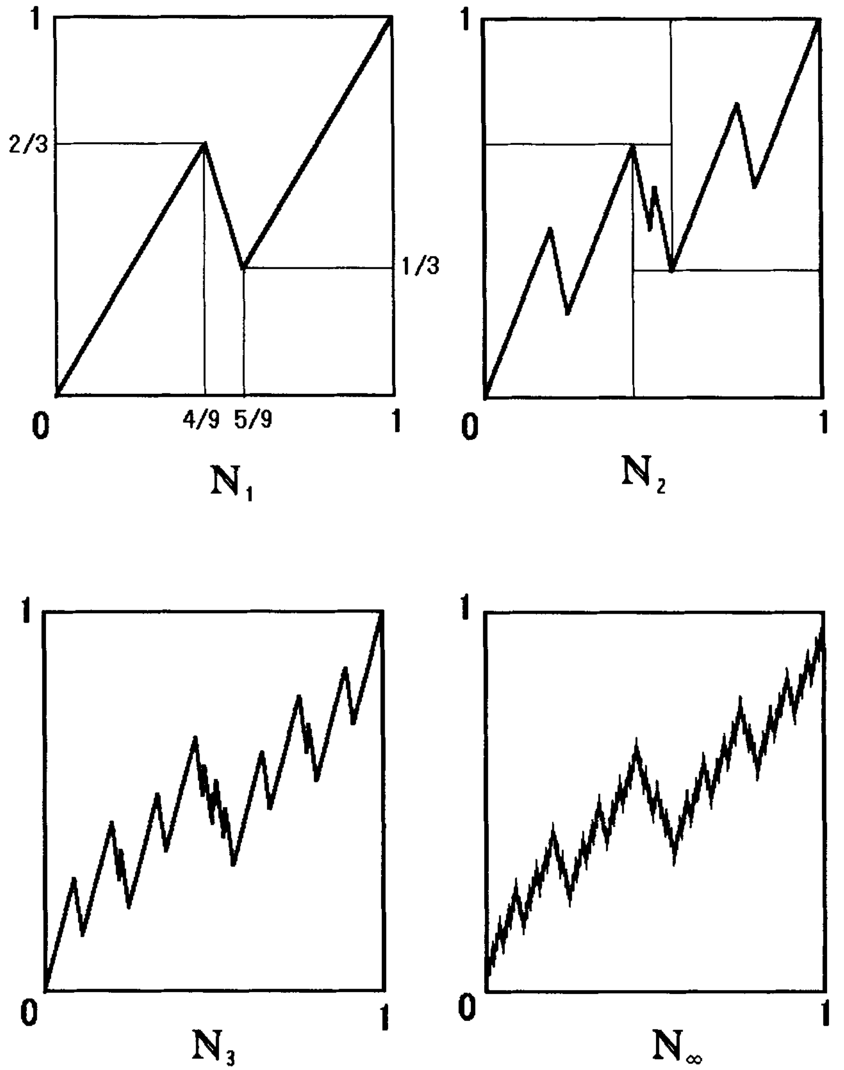

Figure 11. N_1, N_2, N_3 and N_∞

Since there is a one-to-one correspondence between the descendants of R_0 and those of $R_{1,0}$ keeping the color given by

$$[x_1, x_2) \times [y_1, y_2) \to [(4/9)x_1, (4/9)x_2) \times (y_1 + \log(4/9), y_2 + \log(4/9)].$$

Hence by (26),

$$F(\omega_0,(4/9)x_2) - F(\omega_0,(4/9)x_1) = (2/3)(F(\omega_0,x_2) - F(\omega_0,x_1))$$

holds if $[x_1,x_2) \times [y_1,y_2) \in \mathrm{dom}(\omega_0)$ and $[x_1,x_2) \subset [0,1)$. By the continuity of $F(\omega_0,t)$ in t, this implies that

$$F(\omega_0,(4/9)t) = (2/3)F(\omega_0,t)$$

for any $t \in [0,1]$. By the similar correspondence keeping the color between the descendants of R_0 and those of $R_{1,2}$, we have

$$F(\omega_0,(5+4t)/9) - F(\omega_0,5/9) = (2/3)F(\omega_0,t)$$

for any $t \in [0,1]$. By the similar correspondence changing the color between the descendants of R_0 and those of $R_{1,1}$, we have

$$F(\omega_0,(4+t)/9) - F(\omega_0,4/9) = -(1/3)F(\omega_0,t)$$

for any $t \in [0,1]$.

Hence, the graph Γ of the function $F(\omega_0,t)$ on $t \in [0,1]$ satisfies the set equation $\Gamma = \psi(\Gamma)$, where for a compact set R,

$$\psi(R) := \psi_0(R) \cup \psi_1(R) \cup \psi_2(R)$$

with the functions $\psi_0,\psi_1,\psi_2\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$\begin{aligned}
\psi_0(x,y) &= ((4/9)x,(2/3)y),\\
\psi_1(x,y) &= ((x+4)/9,(-y+2)/3),\\
\psi_2(x,y) &= ((4x+5)/9,(2y+1)/3).
\end{aligned}$$

Then Γ is obtained as the limit in the sense of a Hausdorff metric of $\psi^n(\Gamma_{N_0})$ as $n \to \infty$, where Γ_{N_0} denotes the graph of the function $N_0(t) := t(t \in [0,1])$. For $n = 1,2,\ldots$, define a function N_n on $[0,1]$ by $\Gamma_{N_n} = \psi^n(\Gamma_{N_0})$. Then N_1 is a continuous piecewise linear function whose graph consists of 3 line segments,

$$\begin{aligned}
\psi_0(\Gamma_{N_0}) &= \{(x,y); y = (3/2)x0 \le x \le 4/9\},\\
\psi_1(\Gamma_{N_0}) &= \{(x,y); y = -3x+2, 4/9 \le x \le 5/9\},\\
\psi_2(\Gamma_{N_0}) &= \{(x,y); y = (3/2)x-(1/2), 5/9 \le x \le 1\},
\end{aligned}$$

as seen in Figure 11.

Since $\Gamma_{N_2} = \psi(\Gamma_{N_1})$, N_2 is a continuous piecewise linear function on $[0,1]$ obtained by replacing 3 line segments in Γ_{N_1} by self-affine images of Γ_{N_1} or

Γ_{-N_1} keeping the 2 end points fixed. Then, the graph of N_2 consists of 9 line segments. In the same way, we obtain N_n from N_{n-1} for $n = 3, 4, \ldots$. Then, N_n is a continuous piecewise linear function on $[0,1]$ whose graph consists of 3^n line segments.

Let Ξ_n be the set of closed intervals which are the projection to the horizontal axis of the 3^n line segments consisting of the graph of N_n. Note that Ξ_n is the set of closed intervals which are the projection to the vertical axis of the descendents of the n-th generation of the tile R_0 in ω_0. Let Δ_n be the set of the end points of Ξ_n. Let $\Xi = \bigcup_{n=0}^{\infty} \Xi_n$ and $\Delta = \bigcup_{n=0}^{\infty} \Delta_n$. Denote $N_\infty(t) := F(\omega_0, t)$ ($t \in [0,1]$). Then the function $N_\infty(t)$ is the pointwise limit of $N_n(t)$ as $n \to \infty$.

THEOREM 12: (i) *For any s, t with $0 \le s < t \le 1$, $|N_\infty(t) - N_\infty(s)| \le |t-s|^{1/2}$ holds. Moreover, the equality holds if and only if $[s,t] \in \Xi$.*

(ii) *For any Borel set $R \subset [0,1]$,*

$$\int_{N_\infty(t) \in R} dt = \int_R (2 - |4t - 2|) dt.$$

(iii) *The Hausdorff dimension of the graph Γ_{N_∞} of the function N_∞ satisfies $\text{H-}\dim \Gamma_{N_\infty} = 3/2$. In fact, its $(3/2)$-dimensional Hausdorff measure is positive and finite.*

(iv) *$N_\infty(t)$ is locally minimal or maximal at $t = t_0$ if and only if $t_0 \in \Delta$. In this case, there exists $\delta > 0$ such that*

$$(1/\sqrt{5})|h|^{1/2} \le |N_\infty(t_0 + h) - N_\infty(t_0)| \le |h|^{1/2}$$

holds for any h with $t_0 + h \in [0,1]$ and $|h| < \delta$.

(v) *For any $t_0 \in [0,1]$ with $t_0 \notin \Delta$ and $\epsilon > 0$,*

$$\begin{aligned} &\text{H-}\dim\{s \in (t_0 - \epsilon, t_0); N_\infty(s) = N_\infty(t_0)\} \\ &= \text{H-}\dim\{s \in (t_0, t_0 + \epsilon); N_\infty(s) = N_\infty(t_0)\} = 1/2. \end{aligned}$$

In fact, the $(1/2)$-dimensional Hausdorff measures of the above sets are positive and finite.

Proof: (i) is proved in [7].

(ii) Let $\mathcal{L}$ be a bounded operator on the Banach space $C([0,1])$ defined by

$$(\mathcal{L}f)(t) := (4/9)f((2/3)t) + (1/9)f((2-t)/3) + (4/9)f((1+2t)/3)$$

for any $t \in [0,1]$ and $f \in C([0,1])$.

Let ν and τ be the probability measures on $[0,1]$ defined by

$$\nu(S) = \int_{N_\infty(t)\in S} dt \quad \text{and} \quad \int_S d\tau = \int_S (2-|4t-2|)dt$$

for any Borel set $S \subset [0,1]$. We prove that $\nu = \tau$.

For any $f \in C([0,1])$, we have

$$\begin{aligned}
\int \mathcal{L} f d\nu &= \int (\mathcal{L} f)(N_\infty(t))dt \\
&= \int (4/9) f((2/3)N_\infty(t))dt + \int (1/9) f((2-N_\infty(t))/3)dt \\
&\quad + \int (4/9) f((1+2N_\infty(t))/3)dt \\
&= \int (4/9) f(N_\infty((4/9)t))dt + \int (1/9) f(N_\infty((t+4)/9))dt \\
&\quad + \int (4/9) f(N_\infty((4t+5)/9))dt \\
&= \int_0^{4/9} f(N_\infty(t))dt + \int_{4/9}^{5/9} f(N_\infty(t))dt + \int_{5/9}^{1} f(N_\infty(t))dt \\
&= \int f(N_\infty(t))dt = \int f d\nu.
\end{aligned}$$

Thus, ν is invariant under $\mathcal{L}$. We prove that τ is also invariant under $\mathcal{L}$.

In fact,

$$\begin{aligned}
\int \mathcal{L} f d\tau &= \int (2-|4t-2|)(\mathcal{L} f)(t)dt \\
&= \int (2-|4t-2|)(4/9) f((2/3)t)dt + \int (2-|4t-2|)(1/9) f((2-t)/3)dt \\
&\quad + \int (2-|4t-2|)(4/9) f((1+2t)/3)dt \\
&= \int_0^{2/3} (2-|6t-2|)(4/9) f(t)(3/2)dt + \int_{1/3}^{2/3} (2-|12t-6|)(1/9) f(t)3dt \\
&\quad + \int_{1/3}^{1} (2-|6t-4|)(4/9) f(t)(3/2)dt \\
&= \int_0^{1/3} 6t(2/3) f(t)dt \\
&\quad + \int_{1/3}^{1/2} ((4-6t)(2/3) + (-4+12t)(1/3) + (-2+6t)(2/3)) f(t)dt \\
&\quad + \int_{1/2}^{2/3} ((4-6t)(2/3) + (8-12t)(1/3) + (-2+6t)(2/3)) f(t)dt
\end{aligned}$$

$$+\int_{2/3}^{1}(6-6t)(2/3)f(t)dt$$
$$=\int(2-|4t-2|)f(t)dt=\int f d\tau.$$

By the definition of the operator $\mathcal{L}$, we have

$$(\mathcal{L}f)'(t)=(8/27)f'((2/3)t)-(1/27)f'((2-t)/3)+(8/27)f'((1+2t)/3)$$

for any $f\in C^1([0,1])$. This implies that $\|(\mathcal{L}f)'\|\le(17/27)\|f'\|$. Therefore, we have $\|(\mathcal{L}^nf)'\|\le(17/27)^n\|f'\|$, and hence $(\mathcal{L}^nf)'$ converges to 0. This implies that there exists a subsequence $\{n'\}\subset\{n\}$ such that $\mathcal{L}^{n'}f$ converges to a constant, say c. Hence we have

$$\int f d\nu=\lim\int\mathcal{L}^{n'}f d\nu=c=\lim\int\mathcal{L}^{n'}f d\tau=\int f d\tau$$

for any $f\in C^1([0,1])$. This implies that $\nu=\tau$.

(iii) Let $\Gamma:=\Gamma_{N_\infty(t)}$. Since N_∞ is uniformly 1/2-Hölder continuous, the (3/2)-dimensional Hausdorff measure of Γ is finite. We prove that it is positive.

Let ν_Γ be the probability measure supported by Γ such that for any Borel set $S\subset[0,1]$,

$$\nu_\Gamma((S\times[0,1])\cap\Gamma):=\int_S dt.$$

Then by (ii), we have

$$\nu_\Gamma([0,1]\times S)=\int_S(2-|4t-2|)dt. \tag{27}$$

Take any cover $\mathcal{U}=\{U_i;i=1,2,\ldots\}$ of Γ. Let $I_{3/2}(\mathcal{U}):=\sum_i d(U_i)^{3/2}$, where $d(U)$ denotes the diameter of the set U. Since for any set U, we can find a closed rectangle U' such that $U\subset U'$ and $d(U')\le 2\sqrt{2}d(U)$, we can replace each set $U\in\mathcal{U}$ by a closed rectangle like this, so that we have a cover $\mathcal{U}'$ of Γ consisting of closed rectangles such that $I_{3/2}(\mathcal{U}')\le 5I_{3/2}(\mathcal{U})$. Since for any interval $[a,b]\subset[c,d]$, we can find at most 2 intervals I_1,I_2 in Ξ such that $I_1\cup I_2\supset[a,b]$ and $|I_1|+|I_2|\le 9(b-a)$, we replace each rectangle $[a,b]\times[c,d]\in\mathcal{U}'$ as above by $I_1\times[c,d]$ and $I_2\times[c,d]$. Furthermore, we replace them again by $I_i\times([c,d]\cap N_\infty(I_i))(i=1,2)$. Let $\mathcal{V}=\{V_i;i=1,2,\ldots\}$ be the cover of Γ obtained by this procedure from $\mathcal{U}$. Then we have

$$I_{3/2}(\mathcal{V})\le 27I_{3/2}(\mathcal{U}')\le 135I_{3/2}(\mathcal{U}).$$

Take any of $V_i =: I \times [c,d] \in \mathcal{V}$. By the assumption, $I \in \Xi$ and $[c,d] \subset N_\infty(I) =: [C,D]$. Let $N_\infty(I) =: [C,D]$. The graph of Γ restricted to $I \times [C,D]$ is the image of Γ by the mapping $(x,y) \mapsto (a + |I|x, b \pm |I|^{1/2}y)$ with some a,b and $\pm$. Hence by (27), we have

$$\begin{aligned}
\nu_\Gamma(V_i) &= \nu_\Gamma(I \times [c,d]) \\
&= |I|\nu_\Gamma([0,1] \times [\frac{c-C}{D-C}, \frac{d-C}{D-C}]) \\
&\leq 2|I|\frac{x_2 - x_1}{D-C} \\
&= 2|I|\frac{x_2 - x_1}{|I|^{1/2}} \\
&= 2|I|^{1/2}(x_2 - x_1) \leq 2d(V_i)^{3/2}. \qquad (28)
\end{aligned}$$

Thus, adding the above inequality, we have

$$I_{3/2}(\mathcal{V}) = \sum_i d(V_i)^{3/2} \geq (1/2)\sum_i \nu_\Gamma(V_i) \geq (1/2)\nu_\Gamma(\Gamma) = 1/2,$$

so that $I_{3/2}(\mathcal{U}) \geq 1/270$, which completes the proof.

(iv) Let $t_0 \in \Delta \setminus \{0,1\}$. By (i),

$$|N_\infty(t_0 + h) - N_\infty(t_0)| \leq |h|^{1/2}$$

for any h with $t_0 + h \in [0,1]$. Therefore, it is sufficient to prove that there exists $\delta > 0$ such that

$$(1/\sqrt{5})|h|^{1/2} \leq |N_\infty(t_0 + h) - N_\infty(t_0)|$$

for any h with $t_0 + h \in [0,1]$ and $|h| < \delta$. There are intervals I and J in some Γ_n such that $\{t_0\} = I \cap J$ and I is in the left side of J. Then, either the piecewise linear function N_n is increasing in I and decreasing in J or N_n is decreasing in I and increasing in J. Without loss of generality, we assume the latter. Then we have

$$\begin{aligned}
N_\infty(t_0 - |I|s) - N_\infty(t_0) &= |I|^{1/2}(1 - N_\infty(1-s)), \\
N_\infty(t_0 + |J|s) - N_\infty(t_0) &= |J|^{1/2}N_\infty(s),
\end{aligned}$$

for any $s \in [0,1]$, by the set equation $\psi(\Gamma) = \Gamma$. Therefore, with $h = |J|s$,

$$|N_\infty(t_0 + h) - N_\infty(t_0)| \geq \frac{1}{\sqrt{5}}|h|^{1/2}$$

follows from the statement that

$$N_\infty(s) \geq \frac{s^{1/2}}{\sqrt{5}}$$

for any $s \in (0,1]$, and with $h = -|I|s$,

$$|N_\infty(t_0 + h) - N_\infty(t_0)| \geq \frac{1}{\sqrt{5}}|h|^{1/2}$$

follows from the statement that

$$1 - N_\infty(1 - s) \geq \frac{s^{1/2}}{\sqrt{5}}$$

for any $s \in (0,1]$. By the symmetry of the graph of N_∞ with respect to $(1/2, 1/2)$, the second statement follows from the first statement:

$$N_\infty(s) \geq \frac{s^{1/2}}{\sqrt{5}}$$

for any $s \in (0,1]$.

We prove this inequality. Note that the equality holds for $s = 5/9$. Suppose that

$$N_\infty(s)/s^{1/2} < \frac{1}{\sqrt{5}}$$

holds for some $s \in (0,1]$. Since

$$c_0 := \inf_{s\in(0,1]} N_\infty(s)/s^{1/2} = \min_{s\in[4/9,1]} N_\infty(s)/s^{1/2} = \min_{s\in[5/9,1]} N_\infty(s)/s^{1/2},$$

there exists $s_0 \in [5/9, 1]$ which attains the minimum. Then,

$$\begin{aligned} c_0 {s_0}^{1/2} &= N_\infty(s_0) \\ &= (1/3) + N_\infty(s_0 - (5/9)) \\ &> c_0(5/9)^{1/2} + c_0(s_0 - (5/9))^{1/2} \\ &\geq c_0 s_0^{1/2}, \end{aligned}$$

which is a contradiction. Thus, $N_\infty(s) \geq s^{1/2}/\sqrt{5}$ for any $s \in [0,1]$, which completes the proof of (iv).

(v) We prove that for any $x \in (0,1)$, the $(1/2)$-dimensional Hausdorff measure of $(N_\infty)_x$ is positive and finite, where we denote

$$(N_\infty)_x := \{t \in [0,1]; N_\infty(t) = x\}.$$

Let

$$\delta := (2 - |4x - 2|) \wedge (2 - |2(x+1) - 2|) > 0.$$

We prove that for any cover $\mathcal{U}$ of $(N_\infty)_x$, $I_{1/2}(\mathcal{U}) \geq (1/10)\delta$. We may assume that $\mathcal{U}$ consists of open intervals, so that there exists a finite subcover of $\mathcal{U}$.

Therefore, we may assume that $\mathcal{U}$ is a finite cover consisting of closed intervals. Moreover, by the same argument as in the proof of (iii), it is sufficient to prove that for any

$$\mathcal{V} := \{I_i; i = 1, 2, \ldots, K\} \subset \Xi,$$

$I_{1/2}(\mathcal{V}) \geq (1/2)\delta$. Let $\epsilon > 0$ satisfy $x+2\epsilon < 1$ and $|I_i| > \epsilon$ for any $i = 1, 2, \ldots, K$. By the same argument as in (28), we have

$$\nu_\Gamma(I_i \times [x, x+\epsilon]) \leq 2|I_i|\frac{\epsilon}{|I_i|^{1/2}} \leq 2\epsilon|I_i|^{1/2}.$$

Adding the above inequality, we have

$$\nu_\Gamma(\Gamma \cap ([0,1] \times [x, x+\epsilon])) \leq 2\epsilon I_{1/2}(\mathcal{V}).$$

Hence by (27) and the definition of δ, we have

$$I_{1/2}(\mathcal{V}) \geq (2\epsilon)^{-1}\nu_\Gamma([0,1] \times [x, x+\epsilon]) \geq (1/2)\delta,$$

which completes the proof that the Hausdorff measure of $(N_\infty)_x$ is positive.

To prove that it is finite, for any sufficiently small $\epsilon > 0$ and $t \in (N_\infty)_x$, we take $V(t) \in \Xi$ such that $t \in V(t)$ and $2\epsilon \leq |V(t)| < 18\epsilon$. Then there exists a finite subcover $\mathcal{V} := \{V_i : i = 1, 2, \ldots, K\}$ of $\{V(t); t \in (N_\infty)_x\}$ such that $V_i \cap V_j$ is at most one point for any $i \neq j$.

For any $i = 1, 2, \ldots, K$, by the same argument as in (28), we have

$$\begin{aligned}\nu_\Gamma(V_i \times [x - \epsilon^{1/2}, x + \epsilon^{1/2}]) &\geq |V_i| \int_0^{(\epsilon/|V_i|)^{1/2}} (2 - |4t-2|)dt \\ &\geq |V_i| 2\epsilon/|V_i| \geq 2\epsilon \geq (2/5)\epsilon^{1/2}|V_i|^{1/2}.\end{aligned}$$

Adding this inequality together with (27), we have

$$\begin{aligned}4\epsilon^{1/2} &\geq \nu_\Gamma([0,1] \times [x - \epsilon^{1/2}, x + \epsilon^{1/2}]) \\ &= \nu_\Gamma(\Gamma \cap ([0,1] \times [x - \epsilon^{1/2}, x + \epsilon^{1/2}])) \\ &\geq (2/5)\epsilon^{1/2} I_{1/2}(\mathcal{V}).\end{aligned}$$

Hence we have $I_{1/2}(\mathcal{V}) \leq 10$, which completes the proof that the $(1/2)$-dimensional Hausdorff measure of $(N_\infty)_x$ is finite.

The statement in (iv) follows from this. ∎

Consider the stochastic process $(\mathbf{N}_t)_{t \in \mathbb{R}}$ defined by $\mathbf{N}_t(\omega) = F(\omega, t)$, where ω comes from the probability space (Ω, μ), μ being the unique invariant probability measure invariant under the additive action. This process was called the N-process and studied in [7]. A prediction theory based on the N-process was developed. A process $Y_t = H(\mathbf{N}_t, t)$, where the function $H(x,s)$ is an unknown function which is twice continuously differentiable in x and once continuously differentiable in s and $H_x(x,s) > 0$, is considered. The aim is to predict the value Y_c from the observation $Y_J := \{Y_t; t \in J\}$, where $J = [a,b]$ and $a < b < c$.

THEOREM 13 ([7]): *There exists an estimator $\hat{Y}_c$ which is a measurable function of the observation Y_J such that*

$$\mathrm{E}[(\hat{Y}_c - Y_c)^2] = O((c-b)^2)$$

as $c \downarrow b$.

OPEN PROBLEMS: (1) *Does a numeration system which is not a homorphic image of any numeration system coming from weighted substitutions exist? If yes, how to characterize the numeration systems coming from weighted substitutions?*

(2) *Is the condition $B(\sigma,\tau) = \mathbb{R}_+$ necessary for the $\mathbb{R}$-action of a numeration system coming from weighted substitutions with respect to the unique invariant propability measure to be weakly mixing? When does it have a discrete spectrum?*

(3) *What is the multiplicity of the pure Lebesgue spectrum possessed by the $\mathbb{R}$-action of a numeration system coming from a weighted substitution with $B(\sigma,\tau) = \mathbb{R}_+$ with respect to the unique invariant probability measure?*

(4) *When does a numeration system admit an additive group structure consistent with the $(\mathbb{R}, G)$-action?*

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